

## THE KÄHLER-RICCI FLOW ON FANO MANIFOLDS

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## CONTENTS

1. Preliminaries	4
2. The Kähler-Ricci flow and the normalized Kähler-Ricci flow	7
3. The Li-Yau-Hamilton inequalities for KRF	14
4. Perelman's $\mu$ -entropy and $\kappa$ -noncollapsing theorems	20
5. Uniform curvature and diameter estimates for NKRF with nonnegative bisectional curvature	26
6. Perelman's uniform scalar curvature and diameter estimates for NKRF	28
7. Remarks on the formation of singularities in KRF	36
References	41

**Introduction**

In this lecture notes, we aim at giving an introduction to the Kähler-Ricci flow (KRF) on Fano manifolds, i.e., compact Kähler manifolds with positive first Chern class. It will cover mostly some of the developments of the KRF in its first twenty years (1984-2003), especially an essentially self-contained exposition of Perelman's uniform estimates on the scalar curvature, the diameter, and the Ricci potential function (in  $C^1$ -norm) for the normalized Kähler-Ricci flow (NKRF), including the monotonicity of Perelman's  $\mu$ -entropy and  $\kappa$ -noncollapsing theorems for the Ricci flow on compact manifolds. Except in the last section where we shall briefly discuss the formation of singularities of the KRF in Fano case, much of the recent progress since Perelman's uniform estimates are not touched here, especially those by Phong-Sturm [59] and Phong-Song-Sturm-Weinkove [60, 61, 62] (see also [54, 18, 73, 79, 51, 86] etc.) tying the convergence of the NKRF to a notion of GIT stability for the diffeomorphism group, in the spirit of the conjecture of Yau [85] (see also [74, 30]). We hope to discuss these developments, as well as many works related to Kähler-Ricci solitons, on another occasion. We also refer the readers to the recent lecture notes by J. Song and B. Weinkove [71] for some of the other significant developments in KRF.

In spring 1982, Yau invited Richard Hamilton to give a talk at the Institute for Advanced Study (IAS) on his newly completed seminal work "Three-manifolds with positive Ricci curvature" [36]. Shortly after, Yau asked me, Ben Chow and Ngaiming Mok to present Hamilton's work on the Ricci flow in details at Yau's IAS geometry seminar. At the time, Ben Chow and I were first year graduate students, and Mok was an instructor at Princeton University. There was another fellow first

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year graduate student, S. Bando, working with Yau. It was clear to us that Yau was very excited about Hamilton's work and saw its great potential. He encouraged us to study and pursue Hamilton's Ricci flow.

Besides attending courses at Princeton and Yau's lecture series in geometric analysis at IAS, I spent most of 1982 preparing for Princeton's General Examination, a three-hour oral exam covering two basic subjects (Real & Complex Analysis and Algebra) plus two additional advanced topics. But I also continued to study Hamilton's paper. After I passed the General Exam in January 1983, I went to see Yau and asked for his suggestion for a thesis topic. Yau immediately gave me the problem to study the Ricci flow on Kähler manifolds, especially the long time existence and convergence on Fano manifolds. I have to admit that at the time I hardly knew any complex geometry (of course I did not dare to tell Yau so!). In the following months, I spent a lot of time reading and trying to understand Yau's seminal paper on the Calabi conjecture [84], and also Calabi's paper on extremal Kähler metrics [7] suggested by Yau. In the mean time, it happened that Yau invited Calabi to visit IAS in spring 1983 and I benefited a great deal from Calabi's lecture series on "Vanishing theorems in Kähler geometry" at IAS that spring.

By spring 1984 I had managed to prove the long time existence of the canonical Kähler-Ricci flow by adopting Yau's celebrated a priori estimates for the Calabi conjecture to the parabolic case, as well as the convergence to Kähler-Einstein metrics when the first Chern class is either negative or zero, but little progress toward the convergence when first Chern class is positive. Without fully aware of the significance and the difficulties of the problem at the time, I felt kind of embarrassed that I did not meet my adviser's expectation. But to my relief, Yau seemed quite pleased and encouraged me to write up the work. That resulted my 1985 paper [8]. In Fall of 1984, several of Yau's Princeton graduate students, including me and B. Chow, followed him to San Diego where both Richard Hamilton and Rick Schoen also arrived. By then Bando [3] had used the short time property of the flow to classify three-dimensional compact Kähler manifolds of nonnegative bisectional curvature [3] and graduated from Princeton. Shortly after our arrival in San Diego, following Hamilton's work in [37], Ben Chow and I also used the short time property of the flow to classify compact Kähler manifolds with nonnegative curvature operator in all dimensions [15]. In 1988, Mok's work [49] was published in which he was able to show (in 1986) nonnegative bisectional curvature is preserved in all dimensions. By combining the short time property of the flow and the existence of special rational curves by Mori [50], Mok proved the generalized Frankel conjecture in its full generality (see also a recent new proof by H. Gu [34]). Around the same time, Tsuji [80] extended my work on the KRF for the negative Chern class case to compact complex manifolds of general type (see also the related later work of Tian-Zhang [75]). This is a brief history of the KRF in its early years.

Late 1980s and 1990s saw great advances in the Ricci flow by Hamilton [38, 39, 40, 41, 42, 43, 44] which laid the foundation to use the Ricci flow to attack the Poincaré and geometrization conjectures. In particular, the works of Hamilton [38] and Ben Chow [26] imply that every metric on a compact Riemann surface can be deformed to a metric of constant curvature under the Ricci flow. During the same period, there were several developments in the KRF, including the constructions of  $U(n)$ -invariant Kähler-Ricci soliton examples by Koiso [45] and the author [11]<sup>1</sup>;

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<sup>1</sup>My work was carried out at Columbia University in early 1990s.

the Li-Yau-Hamilton inequalities and the Harnack inequality for the KRF [10, 12]; the important work of W.-X. Shi [69, 70], another former student of Yau, using the noncompact KRF to approach Yau's conjecture<sup>2</sup> that a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex Euclidean space  $\mathbb{C}^n$ , etc. In addition, in 1991 at Columbia University, I first observed that Mabuchi's K-energy [48] and the functional defined in Ding-Tian [29] are monotone decreasing under the KRF [9]. The fact that the K-energy is monotone under the KRF turned out to be quite useful, and was first applied in the work of Chen-Tian [23] ten years later.

In November 2002 and spring 2003, Perelman [55, 56, 57] made astounding breakthroughs in the Ricci flow. In April 2003, in a private lecture at MIT, Perelman presented in detail his uniform scalar curvature and diameter estimates for the NKRF based on the monotonicity of his  $\mathcal{W}$ -functional and  $\mu$ -entropy, and the powerful ideas in his  $\kappa$ -noncollapsing results. We remark that prior to Perelman's lecture at MIT, such uniform estimates had appeared only in the important special case when NKRF has positive bisectional curvature, in the work of Chen and Tian [23] for the Kähler surface case (see also [24] for the higher dimensional case) assuming in addition the existence of K-E metrics; and also in the work of B.-L. Chen, X.-P. Zhu and the author [14] in all dimensions and without assuming the existence of K-E metrics.

From Hamilton and Perelman's works to the recent proof of the 1/4-pinching differentiable sphere theorem by Brendle-Schoen [6], we have seen spectacular applications of the Ricci flow and its sheer power of flowing to canonical metrics/structures without a priori knowing their existence. Let us hope to see more such spectacular phenomena happen to the KRF.

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<sup>2</sup>See [21] for a recent survey on the subject.

## 1. PRELIMINARIES

In this section, we fix our notations and recall some basic facts and formulas in Kähler Geometry.

## 1.1 Kähler metrics and Kähler forms

Let  $(X^n, g)$  be a compact Kähler manifold of complex dimension  $n$  with the Kähler metric  $g$ . In local holomorphic coordinates  $(z^1, \dots, z^n)$ , denote its Kähler form by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j. \quad (1.1)$$

By definition,  $g$  is Kähler means that its Kähler form  $\omega$  is a *closed* real (1,1) form, or equivalently,

$$\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}} \quad \text{and} \quad \partial_{\bar{k}} g_{i\bar{j}} = \partial_{\bar{j}} g_{i\bar{k}} \quad (1.2)$$

for all  $i, j, k = 1, \dots, n$ . Here  $\partial_k = \partial/\partial z^k$  and  $\partial_{\bar{k}} = \partial/\partial \bar{z}^k$ .

The cohomology class  $[\omega]$  represented by  $\omega$  in  $H^2(X, \mathbb{R})$  is called the Kähler class of the metric  $g_{i\bar{j}}$ . By the Hodge theory, two Kähler metrics  $g_{i\bar{j}}$  and  $\tilde{g}_{i\bar{j}}$  belong to the same Kähler class if and only if  $g_{i\bar{j}} = \tilde{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$ , or equivalently,

$$\omega = \tilde{\omega} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi \quad (1.3)$$

for some real valued smooth function  $\varphi$  on  $X$ .

The volume of  $(X, g)$  is given by

$$\text{Vol}(X, g) = \int_X \omega^{[n]}, \quad (1.4)$$

where we have followed the convention of Calabi [7] to denote  $\omega^{[n]} = \omega^n/n!$  so that the volume form is given by

$$dV = \det(g_{i\bar{j}}) \wedge_{i=1}^n \left( \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i \right) = \omega^{[n]}. \quad (1.5)$$

Clearly, by Stokes' theorem, if  $g$  and  $\tilde{g}$  are in the same Kähler class then we have

$$\text{Vol}(X, g) = \text{Vol}(X, \tilde{g}).$$

## 1.2 Curvatures and the first Chern class

The Christoffel symbols of the metric  $g_{i\bar{j}}$  are given by

$$\Gamma_{i\bar{j}}^k = g^{k\bar{\ell}} \partial_i g_{j\bar{\ell}} \quad \text{and} \quad \Gamma_{i\bar{j}}^{\bar{k}} = g^{\ell\bar{k}} \partial_{\bar{i}} g_{\ell\bar{j}}, \quad (1.6)$$

where  $(g^{i\bar{j}}) = ((g_{i\bar{j}})^{-1})^T$ . It is a basic fact in Kähler geometry that, for each point  $x_0 \in X^n$ , there exists a system of holomorphic normal coordinates  $(z^1, \dots, z^n)$  at  $x_0$  such that

$$g_{i\bar{j}}(x_0) = \delta_{i\bar{j}} \quad \text{and} \quad \partial_k g_{i\bar{j}}(x_0) = 0, \quad \forall i, j, k = 1, \dots, n. \quad (1.7)$$

The curvature tensor of the metric  $g_{i\bar{j}}$  is defined as  $R_{i\bar{k}\bar{\ell}}^j = -\partial_{\bar{\ell}} \Gamma_{ik}^j$ , or by lowering  $j$  to the second index,

$$R_{i\bar{j}k\bar{\ell}} = g_{p\bar{j}} R_{i\bar{k}\bar{\ell}}^p = -\partial_k \partial_{\bar{\ell}} g_{i\bar{j}} + g^{p\bar{q}} \partial_k g_{i\bar{q}} \partial_{\bar{\ell}} g_{p\bar{j}}. \quad (1.8)$$

From (1.2) and (1.8), we immediately see that  $R_{i\bar{j}k\bar{\ell}}$  is symmetric in  $i$  and  $k$ , in  $\bar{j}$  and  $\bar{\ell}$ , and in the pairs  $\{i\bar{j}\}$  and  $\{k\bar{\ell}\}$ .

We say that  $(X^n, g)$  has positive (holomorphic) bisectional curvature, or positive holomorphic sectional curvature, if

$$R_{i\bar{j}k\bar{\ell}}v^i v^{\bar{j}} w^k w^{\bar{\ell}} > 0, \quad \text{or} \quad R_{i\bar{j}k\bar{\ell}}v^i v^{\bar{j}} v^k v^{\bar{\ell}} > 0$$

respectively, for all nonzero vectors  $v$  and  $w$  in the holomorphic tangent bundle  $T_x X$  of  $X$  at  $x$  for all  $x \in X$ .

The Ricci tensor of the metric  $g_{i\bar{j}}$  is obtained by taking the trace of  $R_{i\bar{j}k\bar{\ell}}$ :

$$R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} = -\partial_i \partial_{\bar{j}} \log \det(g). \quad (1.9)$$

From (1.9), it is clear that the Ricci form

$$Ric = \frac{\sqrt{-1}}{2} \sum_{i,j} R_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad (1.10)$$

is real and closed. It is well known that the first Chern class  $c_1(X) \in H^2(X, \mathbb{Z})$  of  $X$  is represented by the Ricci form:

$$[Ric] = \pi c_1(X). \quad (1.11)$$

Finally, the scalar curvature of the metric  $g_{i\bar{j}}$  is

$$R = g^{i\bar{j}} R_{i\bar{j}}. \quad (1.12)$$

Hence, the total scalar curvature

$$\int_X R dV = \int_X Ric \wedge \omega^{[n-1]}, \quad (1.13)$$

depends only on the Kähler class of  $\omega$  and the first Chern class  $c_1(X)$ .

### 1.3 Covariant derivatives

Given any smooth function  $f$ , we denote by

$$\nabla_i f = \partial_i f, \quad \nabla_{\bar{i}} f = \partial_{\bar{i}} f.$$

For any (1,0)-form  $v_i$ , its covariant derivatives are defined as

$$\nabla_j v_i = \partial_j v_i - \Gamma_{ij}^k v_k \quad \text{and} \quad \nabla_{\bar{j}} v_i = \partial_{\bar{j}} v_i. \quad (1.14)$$

Similarly, for covariant 2-tensors, we have

$$\begin{aligned} \nabla_k v_{i\bar{j}} &= \partial_k v_{i\bar{j}} - \Gamma_{ik}^p v_{p\bar{j}}, & \nabla_{\bar{k}} v_{i\bar{j}} &= \partial_{\bar{k}} v_{i\bar{j}} - \Gamma_{\bar{j}\bar{k}}^{\bar{p}} v_{i\bar{p}}, \\ \nabla_k v_{ij} &= \partial_k v_{ij} - \Gamma_{ik}^p v_{pj} - \Gamma_{jk}^p v_{ip}, & \nabla_{\bar{k}} v_{ij} &= \partial_{\bar{k}} v_{ij}. \end{aligned}$$

Now, in the Kähler case, the second Bianchi identity in Riemannian geometry translates into the relations

$$\nabla_p R_{i\bar{j}k\bar{\ell}} = \nabla_k R_{i\bar{j}p\bar{\ell}} \quad \text{and} \quad \nabla_{\bar{p}} R_{i\bar{j}k\bar{\ell}} = \nabla_{\bar{\ell}} R_{i\bar{j}k\bar{p}}. \quad (1.15)$$

Covariant differentiations of the same type can be commuted freely, e.g.,

$$\nabla_k \nabla_j v_i = \nabla_j \nabla_k v_i, \quad \nabla_{\bar{k}} \nabla_{\bar{j}} v_i = \nabla_{\bar{j}} \nabla_{\bar{k}} v_i, \quad (1.16)$$

etc. But we shall need the following formulas when commuting covariant derivatives of different types:

$$\nabla_k \nabla_{\bar{j}} v_i - \nabla_{\bar{j}} \nabla_k v_i = -R_{k\bar{j}i\bar{\ell}} v_{\bar{\ell}}, \quad (1.17)$$

$$\nabla_k \nabla_{\bar{\ell}} v_{i\bar{j}} - \nabla_{\bar{\ell}} \nabla_k v_{i\bar{j}} = -R_{k\bar{\ell}i\bar{p}} v_{p\bar{j}} + R_{k\bar{\ell}p\bar{j}} v_{i\bar{p}}, \quad (1.18)$$

etc.

We define

$$|\nabla f|^2 = g^{i\bar{j}} \partial_i f \partial_{\bar{j}} f, \quad (1.19)$$

$$|Rc|^2 = g^{i\bar{\ell}} g^{k\bar{j}} R_{i\bar{j}} R_{k\bar{\ell}}, \quad (1.20)$$

and

$$|Rm|^2 = g^{i\bar{q}} g^{p\bar{j}} g^{k\bar{s}} g^{r\bar{\ell}} R_{i\bar{j}k\bar{\ell}} R_{p\bar{q}r\bar{s}}. \quad (1.21)$$

The norm square  $|S|^2$  of any other type of covariant tensor  $S$  is defined similarly.

Finally, the Laplace operator on a tensor  $S$  is, in normal coordinates, defined as

$$\Delta S = \frac{1}{2} \sum_k (\nabla_k \nabla_{\bar{k}} + \nabla_{\bar{k}} \nabla_k) S. \quad (1.22)$$

#### 1.4 Kähler-Einstein metrics and Kähler-Ricci solitons

It is well known that a Kähler metric  $g_{i\bar{j}}$  is Kähler-Einstein if

$$R_{i\bar{j}} = \lambda g_{i\bar{j}}$$

for some real number  $\lambda \in \mathbb{R}$ . Kähler-Ricci solitons are extensions of K-E metrics: a Kähler metric  $g_{i\bar{j}}$  is called a gradient Kähler-Ricci (K-R) soliton if there exists a real-valued smooth function  $f$  on  $X$  such that

$$R_{i\bar{j}} = \lambda g_{i\bar{j}} - \partial_i \partial_{\bar{j}} f \quad \text{and} \quad \nabla_i \nabla_{\bar{j}} f = 0. \quad (1.23)$$

It is called *shrinking* if  $\lambda > 0$ , *steady* if  $\lambda = 0$ , and *expanding* if  $\lambda < 0$ . The function  $f$  is called a potential function.

Note that the second equation in (1.23) is equivalent to saying the gradient vector field

$$\nabla f = (g^{i\bar{j}} \partial_{\bar{j}} f) \frac{\partial}{\partial z^i}$$

is holomorphic. By scaling, we can normalize  $\lambda = 1, 0, -1$  in (1.23). The concept of Ricci soliton was introduced by Hamilton [38] in mid 1980s. It has since played a significant role in Hamilton's Ricci flow as Ricci solitons often arise as singularity models (see, e.g., [13] for a survey). Note that when  $f$  is a constant function, K-R solitons are simply K-E metrics.

Clearly, if  $X^n$  admits a K-E metric or K-R soliton  $g$  then the first Chern class is necessarily definite, as

$$\pi c_1(X) = \lambda [\omega_g].$$

When  $c_1(X) = 0$  it follows from Yau's solution to the Calabi conjecture that in each Kähler class there exists a unique Calabi-Yau metric (i.e., Ricci-flat Kähler metric)  $g$  in that class. Moreover, when  $c_1(X) < 0$ , Aubin [1] and Yau [84] proved independently that there exists a unique Kähler-Einstein metric in the class  $-\pi c_1(X)$ .

However, in the Fano case (i.e.,  $c_1(X) > 0$ ), it is well known that there exist obstructions to the existence of a K-E metric  $g$  in the class of  $\omega \in \pi c_1(X)$  with  $R_{i\bar{j}} = g_{i\bar{j}}$ . One of the obstructions is the Futaki invariant defined as follows: take any Kähler metric  $g$  with  $\omega \in \pi c_1(X)$ . Then its Kähler class  $[\omega]$  agrees with its Ricci class  $[Ric]$ . Hence, by the Hodge theory, there exists a real-valued smooth function  $f$ , called the Ricci potential of the metric  $g$ , such that

$$R_{i\bar{j}} = g_{i\bar{j}} - \partial_i \partial_{\bar{j}} f. \quad (1.24)$$

In [32], Futaki proved that the functional  $F: \eta(X) \rightarrow \mathbb{C}$  defined by

$$F(V) = \int_X \nabla_V f \omega^{[n]} = \int_X (V \cdot \nabla f) \omega^{[n]} \quad (1.25)$$

on the space  $\eta(X)$  of holomorphic vector fields depends only on the class  $\pi c_1(X)$ , but not the metric  $g$ . In particular, if a Fano manifold  $X^n$  admits a positive K-E metric, then the Futaki invariant  $F$  defined above must be zero.

On the other hand, it turns out that compact steady and expanding K-R solitons are necessarily K-E. If  $g$  is a *shrinking* K-R soliton satisfying

$$R_{i\bar{j}} = g_{i\bar{j}} - \partial_i \partial_{\bar{j}} f \quad \text{and} \quad \nabla_i \nabla_{\bar{j}} f = 0 \quad (1.26)$$

with non-constant function  $f$  then, taking  $V = \nabla f$ , we have

$$F(\nabla f) = \int_X |\nabla f|^2 \omega^{[n]} \neq 0. \quad (1.27)$$

The existence of compact shrinking K-R solitons were first shown independently by Koiso [45] and the author [11], and later by X. Wang and X. Zhu [81]. Dancer-Wang [27] extended my construction to the general case when the base manifold is a product of Fano K-E manifolds. The noncompact example shrinking K-R soliton was first found by Feldman-Ilmanen-Knopf [31], see also Dancer-Wang [27] and Futaki-Wang [33] for further examples.

We remark that Bando and Mabuchi [4] proved that positive K-E metrics are unique in the sense that any two positive K-E metrics on  $X^n$  only differ by an automorphism of  $X^n$ . Moreover, Tian and Zhu [77] extended the definition of the Futaki invariant by introducing a corresponding obstruction to the existence of shrinking K-R solitons on Fano manifolds. They also proved the Bando-Mabuchi type uniqueness result for shrinking K-R solitons [76].

## 2. THE KÄHLER-RICCI FLOW AND THE NORMALIZED KÄHLER-RICCI FLOW

In this section we introduce the Kähler-Ricci flow (KRF) and the normalized Kähler-Ricci flow (NKRF) on Fano manifolds, i.e., compact Kähler manifolds with positive first Chern class. We state the basic long time existence of solutions to the NKRF proved by the author in [8], derive the evolution equations of various curvature tensors, and present Mok's result on preserving the non-negativity of the holomorphic bisectional curvature under the KRF.

### 2.1 The Kähler-Ricci flow and the normalized Kähler-Ricci flow

On any given Kähler manifold  $(X^n, \tilde{g}_{i\bar{j}})$ , the Kähler-Ricci flow deforms the initial metric  $\tilde{g}$  by the equation

$$\frac{\partial}{\partial t} g_{i\bar{j}}(t) = -R_{i\bar{j}}(t), \quad g(0) = \tilde{g}, \quad (2.1)$$

or equivalently, in terms of the Kähler forms, by

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)), \quad \omega(0) = \omega_0. \quad (2.1')$$

Note that, by (2.1'), the Kähler class  $[\omega(t)]$  of the evolving metric  $g_{i\bar{j}}(t)$  satisfies the ODE

$$\frac{d}{dt} [\omega(t)] = -\pi c_1(X),$$

from which it follows that

$$[\omega(t)] = [\omega_0] - t\pi c_1(X). \quad (2.2)$$

**Proposition 2.1.** *Given any initial Kähler metric  $\tilde{g}$  on a compact Kähler manifold  $X^n$ , KRF (2.1) admits a unique solution  $g(t)$  for a short time.*

*Proof.* We consider the nonlinear, strictly parabolic, scalar equation of Monge-Ampère type

$$\frac{\partial \varphi}{\partial t} = \log \frac{\det(\tilde{g}_{i\bar{j}} - t\tilde{R}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)}{\det(\tilde{g}_{i\bar{j}})}, \quad \varphi(0) = 0$$

as in [3]. Then, this parabolic equation admits a unique solution  $\varphi$  for a short time, and it is easy to verify that

$$g_{i\bar{j}}(t) =: \tilde{g}_{i\bar{j}} - t\tilde{R}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$$

gives rise to a short time solution to KRF (2.1) for small  $t > 0$ . This proves the existence. For the uniqueness, suppose  $h_{i\bar{j}}$  is another solution to KRF (2.1). Then, by (2.2), we have

$$h_{i\bar{j}} = \tilde{g}_{i\bar{j}} - t\tilde{R}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \psi$$

for some real-valued function  $\psi$ . So it follows that

$$\partial_i \partial_{\bar{j}} \left( \frac{\partial \psi}{\partial t} \right) = -R_{i\bar{j}} + \tilde{R}_{i\bar{j}}.$$

Hence, by (1.9) and by adjusting with an additive function in  $t$  only, we have

$$\frac{\partial \psi}{\partial t} = \log \frac{\det(\tilde{g}_{i\bar{j}} - t\tilde{R}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \psi)}{\det(\tilde{g}_{i\bar{j}})}.$$

Note also that  $h_{i\bar{j}}(0) = \tilde{g}_{i\bar{j}}$  forces  $\psi(0)$  to be a constant function. Therefore  $\varphi$  and  $\psi$  differ by a function in  $t$  only which in turn implies that  $g = h$ .

Alternatively, by the work of Hamilton [36] (see also De Turck [28]), there exists a unique solution  $g(t)$  to (2.1), regarded as the Ricci flow for Riemannian metric, for a short time with  $\tilde{g}$  as the initial metric. Moreover, Hamilton [42] observed that the holonomy group does not change under the Ricci flow for a short time. Thus, the solution  $g(t)$  remains Kähler for  $t > 0$ .  $\square$

**Lemma 2.1.** *Under the Kähler-Ricci flow (2.1), the volume of  $(X, g_{i\bar{j}}(t))$  changes by*

$$\frac{d}{dt} \text{Vol}(X, g(t)) = - \int_X R(t) \omega^{[n]}(t).$$

*Proof.* Under KRF (2.1), we have

$$\frac{\partial}{\partial t} \omega^{[n]} = \left( \frac{\partial}{\partial t} \log \det(g_{i\bar{j}}) \right) \omega^{[n]}$$

and

$$\frac{\partial}{\partial t} \log \det(g_{i\bar{j}}) = g^{i\bar{j}} \frac{\partial}{\partial t} g_{i\bar{j}} = -g^{i\bar{j}} R_{i\bar{j}} = -R.$$

Therefore, the volume element  $dV = \omega^{[n]}$  changes by

$$\frac{\partial}{\partial t} \omega^{[n]} = -R \omega^{[n]}. \tag{2.3}$$

$\square$

From now on, we consider a Fano manifold  $(X^n, \tilde{g}_{i\bar{j}})$  such that

$$[\omega_0] = [\tilde{\omega}] = \pi c_1(X), \tag{2.4}$$

and we deform the initial metric  $\tilde{g}$  by the KRF (2.1).



To keep the volume unchanged, we consider the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}, \quad g(0) = \tilde{g} \quad (2.5)$$

or equivalently

$$\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) + \omega, \quad \omega(0) = \omega_0. \quad (2.5')$$

From the proof of Lemma 2.1, it is easy to see that the following holds (in fact, under NKRF (2.5) the solution  $g(t)$  has the same Kähler class):

**Lemma 2.2.** *Under the normalized Kähler-Ricci flow (2.5), we have*

$$\frac{\partial}{\partial t} (dV) = (n - R) dV.$$

By (2.2) and (2.4), it follows that under the KRF (2.1)

$$[\omega(t)] = \pi(1 - t)c_1(X),$$

showing that  $[\omega(t)]$  shrinks homothetically and would become degenerate at  $t = 1$ . This suggests that if  $[0, T)$  is the maximal existence time interval of solution  $\hat{g}(t)$  to KRF (2.1), then  $T$  cannot exceed 1. We shall see that the NKRF (2.5) has solution  $g(t)$  exists for all time  $0 \leq t < \infty$ , which in turn implies that  $T = 1$  for KRF (2.1).

By direct calculations, one can easily verify the following relations between the solutions to KRF (2.1) and NKRF (2.5).

**Lemma 2.3.** *Let  $\hat{g}_{i\bar{j}}(s)$ ,  $0 \leq s < 1$ , and  $g_{i\bar{j}}(t)$ ,  $0 \leq t < \infty$ , be solutions to the KRF (2.1) and the NKRF (2.5) respectively. Then,  $\hat{g}_{i\bar{j}}(s)$  and  $g_{i\bar{j}}(t)$  are related by*

$$\hat{g}_{i\bar{j}}(s) = (1 - s)g_{i\bar{j}}(t(s)), \quad t = -\log(1 - s)$$

and

$$g_{i\bar{j}}(t) = e^t \hat{g}_{i\bar{j}}(s(t)), \quad s = 1 - e^{-t}.$$

**Corollary 2.1.** *Let  $\hat{g}_{i\bar{j}}(s)$  and  $g_{i\bar{j}}(t)$  be as in Lemma 2.3. Then, their scalar curvatures and the norm square of their curvature tensors are related respectively by*

$$(1 - s)\hat{R}(s) = R(t(s)),$$

and

$$(1 - s)|\hat{Rm}|_{\hat{g}(s)} = |Rm|_{g(t(s))}.$$

## 2.2 The long time existence of the NKRF

First of all, it is well known that the NKRF (2.5) is equivalent to a parabolic scalar equation of complex Monge-Ampère type on the Kähler potential. For any given initial metric  $g_0 = \tilde{g}$  satisfying (2.4), consider

$$g_{i\bar{j}}(t) = \tilde{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi, \quad (2.6)$$

where  $\varphi = \varphi(t)$  is a time-dependent, real-valued, smooth unknown function on  $X$ . Then,

$$\frac{\partial}{\partial t} g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \varphi_t$$

and

$$\begin{aligned} -R_{i\bar{j}} + g_{i\bar{j}} &= -R_{i\bar{j}} + \tilde{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi = -R_{i\bar{j}} + \tilde{R}_{i\bar{j}} + \partial_i \partial_{\bar{j}} (\tilde{f} + \varphi) \\ &= \partial_i \partial_{\bar{j}} \log \frac{\omega^n}{\tilde{\omega}^n} + \partial_i \partial_{\bar{j}} (\tilde{f} + \varphi). \end{aligned}$$

Here  $\tilde{f}$  is the Ricci potential of  $\tilde{g}_{i\bar{j}}$  as defined in (1.24). Thus, the NKRF (2.5) reduces to

$$\partial_i \partial_{\bar{j}} \varphi_t = \partial_i \partial_{\bar{j}} \log \frac{\omega^n}{\tilde{\omega}^n} + \partial_i \partial_{\bar{j}} (\tilde{f} + \varphi),$$

or equivalently,

$$\frac{\partial \varphi}{\partial t} = \log \frac{\det(\tilde{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)}{\det(\tilde{g}_{i\bar{j}})} + \tilde{f} + \varphi + b(t) \quad (2.7)$$

for some function  $b(t)$  of  $t$  only.

Note that (2.7) is strictly parabolic, so standard PDE theory implies its short time existence (cf. [2]). Clearly, we have

**Lemma 2.4.** *If  $\varphi$  solves the parabolic scalar equation (2.7), then  $g_{i\bar{j}}(t)$ , as defined in (2.6), is a solution to the NKRF (2.5).*

Now we can state the following long time existence result shown by the author [8], based on the parabolic version of Yau's a priori estimates in [84]. We refer the readers to [8], or the lecture notes by Song and Weinkove [71] in this volume, for a proof.

**Theorem 2.1** (Cao [8]). *The solution  $\varphi(t)$  to (2.7) exists for all time  $0 \leq t < \infty$ . Consequently, the solution  $g_{i\bar{j}}(t)$  to the normalized Kähler-Ricci flow (2.5) exists for all time  $0 \leq t < \infty$ .*

### 2.3 Preserving positivity of the bisectional curvature

To derive the curvature evolution equations for both KRF and NKRF, we consider

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + \lambda g_{i\bar{j}}. \quad (2.8)$$

**Lemma 2.5.** *Under (2.8), we have*

$$\frac{\partial}{\partial t} R_{i\bar{j}} = \Delta R_{i\bar{j}} + R_{i\bar{j}k\bar{\ell}} R_{\ell\bar{k}} - R_{i\bar{k}} R_{k\bar{j}}, \quad (2.9)$$

and

$$\frac{\partial}{\partial t} R = \Delta R + |Rc|^2 - \lambda R. \quad (2.10)$$

**Proof.** First of all, from (1.9), we get

$$\frac{\partial}{\partial t} R_{i\bar{j}} = -\nabla_i \nabla_{\bar{j}} (g^{k\bar{\ell}} \frac{\partial}{\partial t} g_{k\bar{\ell}}) = \nabla_i \nabla_{\bar{j}} R. \quad (2.11)$$

On the other hand, by using the commuting formulas (1.16)-(1.18) for covariant differentiations, we have

$$\begin{aligned} \nabla_k \nabla_{\bar{k}} R_{i\bar{j}} &= \nabla_k \nabla_{\bar{j}} R_{i\bar{k}} = \nabla_{\bar{j}} \nabla_k R_{i\bar{k}} - R_{k\bar{j}i\bar{\ell}} R_{\ell\bar{k}} + R_{k\bar{j}\ell\bar{k}} R_{i\bar{\ell}} \\ &= \nabla_{\bar{j}} \nabla_i R - R_{i\bar{j}k\bar{\ell}} R_{\ell\bar{k}} + R_{i\bar{\ell}} R_{\ell\bar{j}}, \end{aligned}$$

and

$$\nabla_k \nabla_{\bar{k}} R_{i\bar{j}} = \nabla_{\bar{k}} \nabla_k R_{i\bar{j}}.$$

Hence,

$$\Delta R_{i\bar{j}} = \frac{1}{2} (\nabla_k \nabla_{\bar{k}} + \nabla_{\bar{k}} \nabla_k) R_{i\bar{j}} = \nabla_i \nabla_{\bar{j}} R - R_{i\bar{j}k\bar{\ell}} R_{\ell\bar{k}} + R_{i\bar{\ell}} R_{l\bar{j}}. \quad (2.12)$$

Therefore, (2.9) follows from (2.11) and (2.12)

Next, using the evolution equation of  $R_{i\bar{j}}$ , we have

$$\begin{aligned} \frac{\partial}{\partial t} R &= \frac{\partial}{\partial t} (g^{i\bar{j}} R_{i\bar{j}}) = g^{i\bar{j}} (\Delta R_{i\bar{j}} + R_{i\bar{j}k\bar{\ell}} R_{\ell\bar{k}} - R_{i\bar{k}} R_{k\bar{j}}) + R_{i\bar{j}} (R_{j\bar{i}} - \lambda g_{j\bar{i}}) \\ &= \Delta R + |Rc|^2 - \lambda R. \end{aligned}$$

□

**Lemma 2.6.** *Under (2.8), we have*

$$\begin{aligned} \frac{\partial}{\partial t} R_{i\bar{j}k\bar{\ell}} &= \Delta R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{\ell}} + R_{i\bar{\ell}p\bar{q}} R_{q\bar{p}k\bar{j}} - R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{\ell}} + \lambda R_{i\bar{j}k\bar{\ell}} \\ &\quad - \frac{1}{2} (R_{i\bar{p}} R_{p\bar{j}k\bar{\ell}} + R_{p\bar{j}} R_{i\bar{p}k\bar{\ell}} + R_{k\bar{p}} R_{i\bar{j}p\bar{\ell}} + R_{p\bar{\ell}} R_{i\bar{j}k\bar{p}}). \end{aligned}$$

**Proof.** From (1.8) and by using normal coordinates, we have

$$\begin{aligned} \frac{\partial}{\partial t} R_{i\bar{j}k\bar{\ell}} &= \partial_k \partial_{\bar{\ell}} R_{i\bar{j}} + \lambda R_{i\bar{j}k\bar{\ell}} = \partial_k (\nabla_{\bar{\ell}} R_{i\bar{j}} + \Gamma_{j\bar{\ell}}^{\bar{p}} R_{i\bar{p}}) + \lambda R_{i\bar{j}k\bar{\ell}} \\ &= \nabla_k \nabla_{\bar{\ell}} R_{i\bar{j}} - R_{i\bar{p}} R_{p\bar{j}k\bar{\ell}} + \lambda R_{i\bar{j}k\bar{\ell}}. \end{aligned}$$

On the other hand, by (1.15) and covariant differentiation commuting formulas (1.16)-(1.18), we obtain

$$\nabla_{\bar{p}} \nabla_p R_{i\bar{j}k\bar{\ell}} = \nabla_k \nabla_{\bar{l}} R_{i\bar{j}} - R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{\ell}} + R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{\ell}} - R_{i\bar{\ell}p\bar{q}} R_{q\bar{p}k\bar{j}} + R_{i\bar{j}p\bar{\ell}} R_{k\bar{p}},$$

and

$$\nabla_p \nabla_{\bar{p}} R_{i\bar{j}k\bar{\ell}} = \nabla_{\bar{p}} \nabla_p R_{i\bar{j}k\bar{\ell}} - R_{i\bar{q}} R_{q\bar{j}k\bar{\ell}} + R_{q\bar{j}} R_{i\bar{q}k\bar{\ell}} - R_{k\bar{q}} R_{i\bar{j}q\bar{\ell}} + R_{q\bar{\ell}} R_{i\bar{j}k\bar{q}}.$$

Hence,

$$\begin{aligned} \Delta R_{i\bar{j}k\bar{\ell}} &= \frac{1}{2} (\nabla_p \nabla_{\bar{p}} + \nabla_{\bar{p}} \nabla_p) R_{i\bar{j}k\bar{\ell}} \\ &= \nabla_k \nabla_{\bar{l}} R_{i\bar{j}} - R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{\ell}} + R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{\ell}} - R_{i\bar{\ell}p\bar{q}} R_{q\bar{p}k\bar{j}} \\ &\quad + \frac{1}{2} (-R_{i\bar{p}} R_{p\bar{j}k\bar{\ell}} + R_{p\bar{j}} R_{i\bar{p}k\bar{\ell}} + R_{k\bar{p}} R_{i\bar{j}p\bar{\ell}} + R_{p\bar{\ell}} R_{i\bar{j}k\bar{p}}), \end{aligned}$$

and Lemma 2.6 follows. □

*Remark 2.1.* Clearly, the Ricci evolution equation (2.9) is also a consequence of Lemma 2.6, but the proof in Lemma 2.5 is more direct and easier.

The Ricci flow in general seems to prefer positive curvatures: positive Ricci curvature is preserved in three-dimension [36]; positive scalar curvature, positive curvature operator [37] and positive isotropic curvature [6, 52] are preserved in all dimensions. Here we present a proof of Mok's theorem that positive bisectional curvature is preserved under KRF.

**Theorem 2.2** (Mok [49]). *Let  $(X^n, \tilde{g})$  be a compact Kähler manifold of nonnegative holomorphic bisectional curvature, and let  $g_{i\bar{j}}(t)$  be a solution to the KRF (2.1) or NKRF (2.5) on  $X^n \times [0, T)$ . Then, for  $t > 0$ ,  $g_{i\bar{j}}(t)$  also has nonnegative holomorphic bisectional curvature. Furthermore, if the holomorphic bisectional curvature is positive at one point at  $t = 0$ , then  $g_{i\bar{j}}(t)$  has positive holomorphic bisectional curvature at all points for  $t > 0$ .*

**Proof.** Let us denote by

$$\begin{aligned} F_{i\bar{j}k\bar{\ell}} = & R_{i\bar{j}p\bar{q}}R_{q\bar{p}k\bar{\ell}} - R_{i\bar{p}k\bar{q}}R_{p\bar{j}q\bar{\ell}} + R_{i\bar{\ell}p\bar{q}}R_{q\bar{p}k\bar{j}} + \lambda R_{i\bar{j}k\bar{\ell}} \\ & - \frac{1}{2}(R_{i\bar{p}}R_{p\bar{j}k\bar{\ell}} + R_{p\bar{j}}R_{i\bar{p}k\bar{\ell}} + R_{k\bar{p}}R_{i\bar{j}p\bar{\ell}} + R_{p\bar{\ell}}R_{i\bar{j}k\bar{p}}) \end{aligned}$$

so that

$$\frac{\partial}{\partial t} R_{i\bar{j}k\bar{\ell}} = \Delta R_{i\bar{j}k\bar{\ell}} + F_{i\bar{j}k\bar{\ell}}.$$

By a version of Hamilton's strong tensor maximum principle (cf. [3]), it suffices to show that the following "null-vector condition" holds: for any  $(1, 0)$  vectors  $V = (v^i)$  and  $W = (w^i)$ , we have

$$F_{i\bar{j}k\bar{\ell}} v^i v^{\bar{j}} w^k w^{\bar{\ell}} \geq 0 \quad \text{whenever} \quad R_{i\bar{j}k\bar{\ell}} v^i v^{\bar{j}} w^k w^{\bar{\ell}} = 0, \quad (\text{NVC})$$

or simply,

$$F_{V\bar{V}W\bar{W}} =: F(V, \bar{V}, W, \bar{W}) \geq 0 \quad \text{whenever} \quad R_{V\bar{V}W\bar{W}} =: \text{Rm}(V, \bar{V}, W, \bar{W}) = 0.$$

**Claim 2.1:** If  $R_{V\bar{V}W\bar{W}} = 0$ , then for any  $(1, 0)$  vector  $Z$ , we have

$$R_{V\bar{Z}W\bar{W}} = R_{V\bar{V}W\bar{Z}} = 0.$$

*Proof.* For real parameter  $s \in \mathbb{R}$ , consider

$$G(s) = \text{Rm}(V + sZ, \bar{V} + s\bar{Z}, W, \bar{W}).$$

Since the bisectional curvature is nonnegative and  $R_{V\bar{V}W\bar{W}} = 0$ , it follows that  $G'(0) = 0$  which implies that

$$\text{Re}(R_{V\bar{Z}W\bar{W}}) = 0.$$

Suppose  $R_{V\bar{Z}W\bar{W}} \neq 0$ , and let  $R_{V\bar{Z}W\bar{W}} = |R_{V\bar{Z}W\bar{W}}|e^{\sqrt{-1}\theta}$ . Then, replacing  $Z$  by  $e^{-\sqrt{-1}\theta}Z$  in the above, we get

$$0 = \text{Re}(e^{-\sqrt{-1}\theta}R_{V\bar{Z}W\bar{W}}) = |R_{V\bar{Z}W\bar{W}}|,$$

a contradiction. Thus, we must have

$$R_{V\bar{Z}W\bar{W}} = 0.$$

Similarly, we have  $R_{V\bar{V}W\bar{Z}} = 0$ .

By Claim 2.1, we see that if  $R_{V\bar{V}W\bar{W}} = 0$  then

$$F_{V\bar{V}W\bar{W}} = R_{V\bar{V}Y\bar{Z}}R_{Z\bar{Y}W\bar{W}} - |R_{V\bar{Y}W\bar{Z}}|^2 + |R_{V\bar{W}Y\bar{Z}}|^2.$$

Therefore, (NVC) follows immediately from the following

**Claim 2.2:** Suppose  $R_{V\bar{V}W\bar{W}} = 0$ . Then, for any  $(1, 0)$  vectors  $Y$  and  $Z$ ,

$$R_{V\bar{V}Y\bar{Z}}R_{Z\bar{Y}W\bar{W}} \geq |R_{V\bar{Y}W\bar{Z}}|^2.$$

*Proof.* Consider

$$\begin{aligned} H(s) &= \text{Rm}(V + sY, \bar{V} + s\bar{Y}, W + sZ, \bar{W} + s\bar{Z}) \\ &= s^2 (R_{V\bar{V}ZZ} + R_{Y\bar{Y}W\bar{W}} + R_{V\bar{Y}W\bar{Z}} + R_{Y\bar{V}Z\bar{W}} + R_{V\bar{Y}Z\bar{W}} + R_{Y\bar{V}W\bar{Z}}) + O(s^3). \end{aligned}$$

Here we have used Claim 2.1.

Since  $H(s) \geq 0$  and  $H(0) = 0$ , we have  $H''(0) \geq 0$ . Hence, by taking  $Y = \zeta^k e_k$  and  $Z = \eta^\ell e_\ell$  with respect to any basis  $\{e_1, \dots, e_n\}$ , we obtain a real, semi-positive definite bilinear form  $Q(Y, Z)$ :

$$\begin{aligned} 0 \leq Q(Y, Z) &= R_{V\bar{V}ZZ} + R_{Y\bar{Y}W\bar{W}} + R_{V\bar{Y}W\bar{Z}} + R_{Y\bar{V}Z\bar{W}} + R_{V\bar{Y}Z\bar{W}} + R_{Y\bar{V}W\bar{Z}} \\ &= R_{V\bar{V}e_k\bar{e}_\ell} \eta^k \eta^\ell + R_{e_k\bar{e}_\ell W\bar{W}} \zeta^k \zeta^\ell + R_{V\bar{e}_k W\bar{e}_\ell} \zeta^k \eta^\ell + R_{e_k\bar{V}e_\ell W\bar{W}} \eta^k \eta^\ell \\ &\quad + R_{V\bar{e}_k e_\ell W\bar{W}} \zeta^k \eta^\ell + R_{e_k\bar{V}W\bar{e}_\ell} \zeta^k \eta^\ell \end{aligned}$$

Now we need a useful linear algebra fact (cf. Lemma 4.1 in [10]):

**Lemma 2.7.** *Let  $A$  and  $C$  be two  $m \times m$  real symmetric semi-positive definite matrices, and let  $B$  be a real  $m \times m$  matrix such that the  $2m \times 2m$  real symmetric matrix*

$$G_1 = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

*is semi-positive definite. Then, we have*

$$\text{Tr}(AC) \geq |B|^2.$$

*Proof.* Consider the associated matrix

$$G_2 = \begin{pmatrix} C & -B \\ -B^T & A \end{pmatrix}.$$

It is clear that  $G_2$  is also symmetric and semi-positive definite. Hence, we get

$$\text{Tr}(G_1 G_2) \geq 0.$$

However,

$$G_1 G_2 = \begin{pmatrix} AC - BB^T & BA - AB \\ B^T C - CB^T & CA - B^T B \end{pmatrix}.$$

Therefore,

$$\text{Tr}(AC) - |B|^2 = \frac{1}{2} \text{Tr}(G_1 G_2) \geq 0.$$

Now, by applying Lemma 2.7 to the above semi-positive definite real bilinear form  $Q$ , one gets

$$R_{V\bar{V}Y\bar{Z}} R_{Z\bar{Z}W\bar{W}} \geq |R_{V\bar{Y}W\bar{Z}}|^2 + |R_{V\bar{W}Y\bar{Z}}|^2.$$

□

*Remark 2.2.* S. Bando [3] first proved Theorem 2.2 for  $n = 3$ , and W. -X. Shi [70] extended Theorem 2.2 to the complete noncompact case with bounded curvature tensor.

*Remark 2.3.* By using a certain special evolving orthonormal frame  $\{e_\alpha\}$  under KRF (2.1) similarly as in [37] (see also Section 5 of [70]), one obtains the simplified evolution equation

$$\frac{\partial}{\partial t} R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \Delta R_{\alpha\bar{\beta}\gamma\bar{\delta}} + R_{\alpha\bar{\beta}\mu\bar{\nu}} R_{\nu\bar{\mu}\gamma\bar{\delta}} + R_{\alpha\bar{\delta}\mu\bar{\nu}} R_{\nu\bar{\mu}\gamma\bar{\beta}} - R_{\alpha\bar{\mu}\gamma\bar{\nu}} R_{\mu\bar{\beta}\nu\bar{\delta}} + \lambda R_{\alpha\bar{\beta}\gamma\bar{\delta}}, \quad (2.13)$$

where  $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$  is the Riemannian curvature tensor components with respect to the evolving frame  $\{e_\alpha\}$ . By using (2.13) and modifying the above proof of Theorem 2.2, one can prove that nonnegative *orthogonal bisectional curvature*,  $Rm(V, \bar{V}, W, \bar{W}) \geq 0$  with  $V \perp W$ , is also preserved under KRF (2.1), a fact first observed by Hamilton and the author [16] in 1992 at IAS (see also [35]).

*Remark 2.4.* Wilking [82] has provided a nice Lie algebra approach treating all known nonnegativity curvature conditions preserved under the Ricci flow and KRF so far, including the nonnegative bisectional curvature and the nonnegative orthogonal bisectional curvature.

### 3. THE LI-YAU-HAMILTON INEQUALITIES FOR KRF

In [47], Li-Yau developed a fundamental gradient estimate, now called Li-Yau estimate (aka differential Harnack inequality), for positive solutions to the heat equation on a complete Riemannian manifold with nonnegative Ricci curvature. They used it to derive the Harnack inequality for such solutions by a path integration. Shortly after, based on a suggestion of Yau, Hamilton [38] derived a similar estimate for the scalar curvature of solutions to the Ricci flow on Riemann surfaces with positive curvature. Hamilton subsequently found a matrix version of the Li-Yau estimate [39] for solutions to the Ricci flow with positive curvature operator in all dimensions. This matrix version of the Li-Yau estimate is now called **Li-Yau-Hamilton estimate**, and it played a central role in the analysis of formation of singularities and the application of the Ricci flow to three-manifold topology. Around the same time, the author obtained the (matrix) Li-Yau-Hamilton estimate for the KRF with nonnegative bisectional curvature and the Harnack inequality for the evolving scalar curvature, as well as the determinant of the Ricci tensor, by a similar path integration argument. We remark that our Li-Yau-Hamilton estimate for the KRF in the noncompact case played a crucial role in the works of Chen-Tang-Zhu CTZ04, Ni [53], Chau-Tam[20], etc. The presentation below essentially follows the original papers of Hamilton [38, 39, 40] and the author [10, 12].

We shall start by recalling the well-known Li-Yau inequality for positive solutions to the heat equation on complete Riemannian manifolds with nonnegative Ricci curvature, and the important observation that Li-Yau inequality becomes equality on the standard heat kernel on the Euclidean space. Then, following Hamilton, we show how one could derive the matrix Li-Yau-Hamilton quadratic for the KRF from the equation of expanding Kähler-Ricci solitons. Finally we state and sketch the matrix Li-Yau-Hamilton inequality for the KRF with nonnegative bisectional curvature.

#### 3.1 The Li-Yau estimate for the 2-dimensional Ricci flow

Let us begin by describing the Li-Yau estimate [47] for positive solutions to the heat equation on a complete Riemannian manifold with nonnegative Ricci curvature.

**Theorem 3.1** (Li-Yau [47]). *Let  $(M, g_{ij})$  be an  $n$ -dimensional complete Riemannian manifold with nonnegative Ricci curvature. Let  $u(x, t)$  be any positive solution to the heat equation*

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{on } M \times [0, \infty).$$

Then, for all  $t > 0$ , we have

$$\frac{\partial u}{\partial t} - \frac{|\nabla u|^2}{u} + \frac{n}{2t}u \geq 0 \quad \text{on } M \times (0, \infty). \quad (3.1)$$

We remark that, as observed by Hamilton (cf. [39]), one can in fact prove that for any vector field  $V^i$  on  $M$ ,

$$\frac{\partial u}{\partial t} + 2\nabla u \cdot V + u|V|^2 + \frac{n}{2t}u \geq 0. \quad (3.2)$$

If we take the optimal vector field  $V = -\nabla u/u$ , then we recover the inequality (3.1).

Now we consider the Ricci flow on a Riemann surface. Since in (real) dimension two the Ricci curvature is given by

$$R_{ij} = \frac{1}{2}Rg_{ij},$$

the Ricci flow becomes

$$\frac{\partial g_{ij}}{\partial t} = -Rg_{ij}. \quad (3.3)$$

Now let  $g_{ij}(t)$  be a complete solution of the Ricci flow (3.3) on a Riemann surface  $M$  and  $0 \leq t < T$ . Then the scalar curvature  $R$  evolves by the semilinear equation

$$\frac{\partial R}{\partial t} = \Delta R + R^2$$

on  $M \times [0, T)$ . Suppose the scalar curvature of the initial metric is bounded, non-negative everywhere and positive somewhere. Then it follows from the maximum principle that the scalar curvature  $R(x, t)$  of the evolving metric remains nonnegative. Moreover, from the standard strong maximum principle (which works in each local coordinate neighborhood), the scalar curvature is positive everywhere for  $t > 0$ . In [38], Hamilton obtained the following Li-Yau estimate for the scalar curvature  $R(x, t)$ .

**Theorem 3.2** (Hamilton [38]). *Let  $g_{ij}(t)$  be a complete solution to the Ricci flow on a surface  $M$ . Assume the scalar curvature of the initial metric is bounded, nonnegative everywhere and positive somewhere. Then the scalar curvature  $R(x, t)$  satisfies the Li-Yau estimate*

$$\frac{\partial R}{\partial t} - \frac{|\nabla R|^2}{R} + \frac{R}{t} \geq 0. \quad (3.4)$$

**Proof.** By the above discussion, we know  $R(x, t) > 0$  for  $t > 0$ . If we set

$$L = \log R(x, t) \quad \text{for } t > 0,$$

then

$$\begin{aligned} \frac{\partial}{\partial t} L &= \frac{1}{R}(\Delta R + R^2) \\ &= \Delta L + |\nabla L|^2 + R \end{aligned}$$

and (3.4) is equivalent to

$$\frac{\partial L}{\partial t} - |\nabla L|^2 + \frac{1}{t} = \Delta L + R + \frac{1}{t} \geq 0.$$

Following Li-Yau [47] in the linear heat equation case, we consider the quantity

$$Q = \frac{\partial L}{\partial t} - |\nabla L|^2 = \Delta L + R.$$

Then by a direct computation,

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \frac{\partial}{\partial t}(\Delta L + R) \\ &= \Delta \left( \frac{\partial L}{\partial t} \right) + R \Delta L + \frac{\partial R}{\partial t} \\ &= \Delta Q + 2 \nabla L \cdot \nabla Q + 2 |\nabla^2 L|^2 + 2R(\Delta L) + R^2 \\ &\geq \Delta Q + 2 \nabla L \cdot \nabla Q + Q^2. \end{aligned}$$

So we get

$$\frac{\partial}{\partial t} \left( Q + \frac{1}{t} \right) \geq \Delta \left( Q + \frac{1}{t} \right) + 2 \nabla L \cdot \nabla \left( Q + \frac{1}{t} \right) + \left( Q - \frac{1}{t} \right) \left( Q + \frac{1}{t} \right).$$

Hence by the maximum principle argument, we obtain

$$Q + \frac{1}{t} \geq 0.$$

This proves the theorem.  $\square$

### 3.2 Li-Yau estimate and expanding solitons

To prove inequality (3.4) for the scalar curvature of solutions to the Ricci flow in higher dimensions is not so simple. It turns out that one does not get inequality (3.4) directly, but rather indirectly as the trace of certain matrix estimate when either curvature operator (in the Riemannian case) or bisectional curvature (in the Kähler case) is nonnegative. The key ingredient in formulating this matrix version is an important observation by Hamilton that the Li-Yau inequality, as well as its matrix version, becomes equality on the expanding solitons which he first discovered for the case of the heat equation on  $\mathbb{R}^n$ . This led him and the author to formulate and prove the matrix differential Harnack inequality, now called Li-Yau-Hamilton estimates, for the Ricci flow in higher dimensions [39, 40] and the Kähler-Ricci flow [10, 12] respectively.

To illustrate, let us examine the heat equation case first. Consider the heat kernel

$$u(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \quad (3.5)$$

which can be considered as an expanding soliton solution for the standard heat equation on  $\mathbb{R}^n$ .

Differentiating the function  $u$  once, we get

$$\nabla_j u = -u \frac{x_j}{2t} \quad \text{or} \quad \nabla_j u + u V_j = 0, \quad (3.6)$$

where

$$V_j = \frac{x_j}{2t} = -\frac{\nabla_j u}{u}.$$

Differentiating (3.6) again, we have

$$\nabla_i \nabla_j u + \nabla_i u V_j + \frac{u}{2t} \delta_{ij} = 0. \quad (3.7)$$



To make the expression in (3.7) symmetric in  $i, j$ , we multiply  $V_i$  to (3.6) and add to (3.7) and obtain

$$\nabla_i \nabla_j u + \nabla_i u V_j + \nabla_j u V_i + u V_i V_j + \frac{u}{2t} \delta_{ij} = 0. \quad (3.8)$$

Taking the trace in (3.8) and using the equation  $\partial u / \partial t = \Delta u$ , we arrive at

$$\frac{\partial u}{\partial t} + 2 \nabla u \cdot V + u |V|^2 + \frac{n}{2t} u = 0,$$

which shows that the Li-Yau inequality (3.1) becomes an equality on our expanding soliton solution  $u$ ! Moreover, we even have the matrix identity (3.8).

Based on the above observation and in a similar process, Hamilton [39] found a matrix quantity, which vanishes on expanding gradient Ricci solitons and is non-negative for any solution to the Ricci flow with nonnegative curvature operator. At the same time, the author [10] (see also [12]) proved the Li-Yau-Hamilton estimate for the Kähler-Ricci flow with nonnegative bisectional curvature, see below.

To formulate the Li-Yau-Hamilton quadric, let us consider a homothetically expanding gradient Kähler-Ricci soliton  $g$  satisfying

$$R_{i\bar{j}} + \frac{1}{t} g_{i\bar{j}} = \nabla_i V_{\bar{j}}, \quad \nabla_i V_j = 0 \quad (3.9)$$

with  $V_i = \nabla_i f$  for some real-valued smooth function  $f$  on  $X$ . Differentiating (3.9) and commuting give the first order relations

$$\nabla_k R_{i\bar{j}} = \nabla_k \nabla_{\bar{j}} V_i - \nabla_{\bar{j}} \nabla_k V_i = -R_{k\bar{j}i\bar{p}} V_p,$$

or

$$\nabla_k R_{i\bar{j}} + R_{i\bar{j}k\bar{p}} V_p = 0, \quad (3.10)$$

and

$$\nabla_k R_{i\bar{j}} V_{\bar{k}} + R_{i\bar{j}k\bar{p}} V_p V_{\bar{k}} = 0. \quad (3.11)$$

Differentiating (3.10) again and using the first equation in (3.9), we get

$$\nabla_{\bar{l}} \nabla_k R_{i\bar{j}} + \nabla_{\bar{p}} R_{i\bar{j}k\bar{l}} V_p + R_{i\bar{j}k\bar{p}} R_{p\bar{l}} + \frac{1}{t} R_{i\bar{j}k\bar{l}} = 0. \quad (3.12)$$

Taking the trace in (3.12), we get

$$\Delta R_{i\bar{j}} + \nabla_{\bar{k}} R_{i\bar{j}} V_k + R_{i\bar{j}k\bar{l}} R_{l\bar{k}} + \frac{1}{t} R_{i\bar{j}} = 0. \quad (3.13)$$

Symmetrizing by adding (3.11) to (3.13), we arrive at

$$\Delta R_{i\bar{j}} + \nabla_k R_{i\bar{j}} V_{\bar{k}} + \nabla_{\bar{k}} R_{i\bar{j}} V_k + R_{i\bar{j}k\bar{l}} R_{l\bar{k}} + R_{i\bar{j}k\bar{l}} V_l V_{\bar{k}} + \frac{1}{t} R_{i\bar{j}} = 0,$$

or, by (2.9), equivalently

$$\frac{\partial}{\partial t} R_{i\bar{j}} + \nabla_k R_{i\bar{j}} V_{\bar{k}} + \nabla_{\bar{k}} R_{i\bar{j}} V_k + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j}k\bar{l}} V_l V_{\bar{k}} + \frac{1}{t} R_{i\bar{j}} = 0. \quad (3.14)$$

### 3.3 The Li-Yau-Hamilton estimates and Harnack's inequalities for KRF

We now state the Li-Yau-Hamilton estimates and the Harnack inequalities for KRF and NKRF with nonnegative holomorphic bisectional curvature.

**Theorem 3.3** (Cao [10, 12]). *Let  $g_{i\bar{j}}(t)$  be a complete solution to the Kähler-Ricci flow on  $X^n$  with bounded curvature and nonnegative bisectional curvature and  $0 \leq t < T$ . For any point  $x \in X$  and any vector  $V$  in the holomorphic tangent space  $T_x^{1,0}X$ , let*

$$Z_{i\bar{j}} = \frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + \nabla_k R_{i\bar{j}} V^k + \nabla_{\bar{k}} R_{i\bar{j}} V^{\bar{k}} + R_{i\bar{j}k\bar{\ell}} V^k V^{\bar{\ell}} + \frac{1}{t} R_{i\bar{j}}.$$

Then we have

$$Z_{i\bar{j}} W^i W^{\bar{j}} \geq 0$$

for all  $x \in X$ ,  $V, W \in T_x^{1,0}X$ , and  $t > 0$ .

The proof of Theorem 3.3 is based on Hamilton's strong tensor maximum principle and involves a large amount of calculations. We refer the interested reader to the original papers [10, 12] for details.

**Corollary 3.1** (Cao [10, 12]). *Under the assumptions of Theorem 3.3, the scalar curvature  $R$  satisfies the estimate*

$$\frac{\partial R}{\partial t} + \nabla_i R V^i + \nabla_{\bar{i}} R V^{\bar{i}} + R_{i\bar{j}} V^i V^{\bar{j}} + \frac{R}{t} \geq 0 \quad (3.15)$$

for all  $x \in X$  and  $t > 0$ . In particular,

$$\frac{\partial R}{\partial t} - \frac{|\nabla R|^2}{R} + \frac{R}{t} \geq 0. \quad (3.16)$$

**Proof.** The first inequality (3.15) follows by taking the trace of  $Z_{i\bar{j}}$  in Theorem 3.3. By taking  $V = -\nabla \log R$  in (3.15) and observing  $R_{i\bar{j}} \leq R g_{i\bar{j}}$  (because  $R_{i\bar{j}} \geq 0$ ), we obtain the second inequality (3.16).  $\square$

As a consequence of Corollary 3.1, we obtain the following Harnack inequality for the scalar curvature  $R$  by taking the Li-Yau type path integral as in [47].

**Corollary 3.2** (Cao [10, 12]). *Let  $g_{i\bar{j}}(t)$  be a complete solution to the Kähler-Ricci flow on  $X^n$  with bounded and nonnegative bisectional curvature. Then for any points  $x_1, x_2 \in X$ , and  $0 < t_1 < t_2$ , we have*

$$R(x_1, t_1) \leq \frac{t_2}{t_1} e^{d_{t_1}(x_1, x_2)^2 / 4(t_2 - t_1)} R(x_2, t_2).$$

Here  $d_{t_1}(x_1, x_2)$  denotes the distance between  $x_1$  and  $x_2$  with respect to  $g_{i\bar{j}}(t_1)$ .

**Proof.** Take the geodesic path  $\gamma(\tau)$ ,  $\tau \in [t_1, t_2]$ , from  $x_1$  to  $x_2$  at time  $t_1$  with constant velocity  $d_{t_1}(x_1, x_2)/(t_2 - t_1)$ . Consider the space-time path  $\eta(\tau) = (\gamma(\tau), \tau)$ ,  $\tau \in [t_1, t_2]$ . We compute

$$\begin{aligned} \log \frac{R(x_2, t_2)}{R(x_1, t_1)} &= \int_{t_1}^{t_2} \frac{d}{d\tau} \log R(\gamma(\tau), \tau) d\tau \\ &= \int_{t_1}^{t_2} \frac{1}{R} \left( \frac{\partial R}{\partial \tau} + \nabla R \cdot \frac{d\gamma}{d\tau} \right) d\tau \\ &\geq \int_{t_1}^{t_2} \left( \frac{\partial \log R}{\partial \tau} - |\nabla \log R|_{g(\tau)}^2 - \frac{1}{4} \left| \frac{d\gamma}{d\tau} \right|_{g(\tau)}^2 \right) d\tau. \end{aligned}$$

Then, by the Li-Yau estimate (3.16) for  $R$  in Corollary 3.1 and the fact that the metric is shrinking (since the Ricci curvature is nonnegative), we have

$$\begin{aligned} \log \frac{R(x_2, t_2)}{R(x_1, t_1)} &\geq \int_{t_1}^{t_2} \left( -\frac{1}{\tau} - \frac{1}{4} \left| \frac{d\gamma}{d\tau} \right|_{g(t_1)}^2 \right) d\tau \\ &= \log \frac{t_1}{t_2} - \frac{d_{t_1}(x_1, x_2)^2}{4(t_2 - t_1)}. \end{aligned}$$

Now the desired Harnack inequality follows by exponentiating.  $\square$

Finally, we can convert Corollary 3.1 and Corollary 3.2 to the NKRF case and yield the following Li-Yau type estimate and Harnack's inequality.

**Theorem 3.4** (Cao [10]). *Let  $g_{i\bar{j}}(t)$  be a solution to NKRF on  $X^n \times [0, \infty)$  with nonnegative bisectional curvature. Then, the scalar curvature  $R$  satisfies*

(a) *the Li-Yau type estimate: for any  $t > 0$  and  $x \in X$ ,*

$$\frac{\partial R}{\partial t} - \frac{|\nabla R|^2}{R} + \frac{R}{1 - e^{-t}} \geq 0; \quad (3.17)$$

(b) *the Harnack inequality: for any  $0 < t_1 < t_2$  and any  $x, y \in X$ ,*

$$R(x, t_1) \leq \frac{e^{t_2} - 1}{e^{t_1} - 1} \exp\{e^{t_2 - t_1} \frac{d_{t_1}^2(x, y)}{4(t_2 - t_1)}\} R(y, t_2), \quad (3.18)$$

**Proof.** Part (a): Let  $\hat{g}_{i\bar{j}}(s)$  be the associated solution to KRF on  $X \times [0, 1)$ . By Lemma 2.3, Corollary 2.1 and Corollary 3.1, we have

$$R = (1 - s)\hat{R}, \quad 1 - e^{-t} = s,$$

and

$$\frac{\partial \hat{R}}{\partial s} - \frac{|\nabla \hat{R}|_{\hat{g}}^2}{\hat{R}} + \frac{\hat{R}}{s} \geq 0.$$

It is then easy to check that they are translated into (3.17).

Part (b): By the Li-Yau path integration argument as in the proof of Corollary 3.2 but use (3.17) instead, we get

$$\begin{aligned} \log \frac{R(y, t_2)}{R(x, t_1)} &\geq \int_{t_1}^{t_2} \left( -\frac{1}{1 - e^{-\tau}} - \frac{1}{4} \left| \frac{d\gamma}{d\tau} \right|_{g(\tau)}^2 \right) d\tau \\ &= \log \frac{e^{t_1} - 1}{e^{t_2} - 1} - \frac{1}{4} \Delta(x, t_1; y, t_2). \end{aligned}$$

where

$$\Delta(x, t_1; y, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} |\gamma'(\tau)|_{g(\tau)}^2 d\tau. \quad (3.19)$$

But, the NKRF equation and the assumption of  $Rc_g \geq 0$  imply that, for  $t_1 < t_2$ ,

$$g(t_2) \leq e^{t_2 - t_1} g(t_1).$$

Hence,

$$\Delta(x, t_1; y, t_2) \leq e^{t_2 - t_1} \frac{d_{t_1}^2(x, y)}{(t_2 - t_1)}.$$

Therefore,

$$\log \frac{R(y, t_2)}{R(x, t_1)} \geq \log \frac{e^{t_1} - 1}{e^{t_2} - 1} - e^{t_2 - t_1} \frac{d_{t_1}^2(x, y)}{4(t_2 - t_1)}.$$

□

#### 4. PERELMAN'S $\mu$ -ENTROPY AND $\kappa$ -NONCOLLAPSING THEOREMS

In this section, we review Perelman's  $\mathcal{W}$ -functional and the associated  $\mu$ -entropy. We show that the  $\mu$ -entropy is monotone under the Ricci flow and use this important fact to prove a strong  $\kappa$ -noncollapsing theorem for the Ricci flow on compact Riemannian manifolds. These results and the ideas in the proof play a crucial role in the next two sections when we discuss the uniform estimates on the diameter and the scalar curvature of the NKFR.

##### 4.1 Perelman's $\mathcal{W}$ -functional and $\mu$ -entropy for the Ricci flow

Let  $M$  be a compact  $n$ -dimensional manifold. Consider the following functional, due to Perelman [55],

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \quad (4.1)$$

under the constraint

$$(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} dV = 1. \quad (4.2)$$

Here  $g_{ij}$  is any given Riemannian metric,  $f$  is any smooth function on  $M$ , and  $\tau$  is a positive scale parameter. Clearly the functional  $\mathcal{W}$  is invariant under simultaneous scaling of  $\tau$  and  $g_{ij}$  (or equivalently the parabolic scaling), and invariant under diffeomorphism. Namely, for any positive number  $a > 0$  and any diffeomorphism  $\varphi \in \text{Diff}(M^n)$ ,

$$\mathcal{W}(\varphi^* g_{ij}, \varphi^* f, \tau) = \mathcal{W}(g_{ij}, f, \tau) \quad \text{and} \quad \mathcal{W}(ag_{ij}, f, a\tau) = \mathcal{W}(g_{ij}, f, \tau). \quad (4.3)$$

In [55] Perelman derived the following first variation formula (see also [19])

**Lemma 4.1** (Perelman [55]). *If  $v_{ij} = \delta g_{ij}$ ,  $h = \delta f$ , and  $\eta = \delta\tau$ , then*

$$\begin{aligned} & \delta\mathcal{W}(v_{ij}, h, \eta) \\ &= \int_M -\tau v_{ij} \left( R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right) (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\ &+ \int_M \left( \frac{v}{2} - h - \frac{n}{2\tau} \eta \right) [\tau(R + 2\Delta f - |\nabla f|^2) + f - n - 1] (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\ &+ \int_M \eta \left( R + |\nabla f|^2 - \frac{n}{2\tau} \right) (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV. \end{aligned}$$

Here  $v = g^{ij} v_{ij}$ .

By Lemma 4.1 and direct computations (cf. [55, 19]), one obtains

**Lemma 4.2** (Perelman [55]). *If  $g_{ij}(t)$ ,  $f(t)$  and  $\tau(t)$  evolve according to the system*

$$\begin{cases} \frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \\ \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\ \frac{\partial \tau}{\partial t} = -1, \end{cases}$$

then

$$\frac{d}{dt} \mathcal{W}(g_{ij}(t), f(t), \tau(t)) = \int_M 2\tau \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV,$$

and  $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$  is constant. In particular  $\mathcal{W}(g_{ij}(t), f(t), \tau(t))$  is nondecreasing in time and the monotonicity is strict unless we are on a shrinking gradient soliton.

Now we define

$$\mu(g_{ij}, \tau) = \inf \left\{ \mathcal{W}(g_{ij}, f, \tau) \mid f \in C^\infty(M), \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} dV = 1 \right\}. \quad (4.4)$$

Note that if we set  $u = e^{-f/2}$ , then the functional  $\mathcal{W}$  can be expressed as

$$\mathcal{W} = \mathcal{W}(g_{ij}, u, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2] dV \quad (4.5)$$

and the constraint (4.2) becomes

$$(4\pi\tau)^{-\frac{n}{2}} \int_M u^2 dV = 1. \quad (4.6)$$

Thus  $\mu(g_{ij}, \tau)$  corresponds to the best constant of a logarithmic Sobolev inequality. Since the non-quadratic term is subcritical (in view of Sobolev exponent), it is rather straightforward to show that

$$\inf \left\{ (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(4|\nabla u|^2 + Ru^2) - u^2 \log u^2 - nu^2] dV : (4\pi\tau)^{-\frac{n}{2}} \int_M u^2 dV = 1 \right\}$$

is achieved by some nonnegative function  $u \in H^1(M)$  which satisfies the Euler-Lagrange equation

$$\tau(-4\Delta u + Ru) - 2u \log u - nu = \mu(g_{ij}, \tau)u.$$

One can further show that  $u$  is positive (see [63]). Then the standard regularity theory of elliptic PDEs shows that  $u$  is smooth. We refer the reader to Rothaus [63] for more details. It follows that  $\mu(g_{ij}, \tau)$  is achieved by a minimizer  $f$  satisfying the nonlinear equation

$$\tau(2\Delta f - |\nabla f|^2 + R) + f - n = \mu(g_{ij}, \tau). \quad (4.7)$$

It turns out that the  $\mu$ -entropy has the following important monotonicity property under the Ricci flow:

**Proposition 4.1** (Perelman [55]). *Let  $g_{ij}(t)$  be a solution to the Ricci flow*

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

*on  $M^n \times [0, T)$  with  $0 < T < \infty$ , then  $\mu(g_{ij}(t), T_0 - t)$  is nondecreasing along the Ricci flow for any  $T_0 \geq T$ ; moreover, the monotonicity is strict unless we are on a shrinking gradient soliton.*

**Proof.** Fix any time  $t_0$ , let  $f_0$  be a minimizer of  $\mu(g_{ij}(t_0), T_0 - t_0)$ . Note that the backward heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}$$

is equivalent to the linear equation

$$\frac{\partial}{\partial t}((4\pi\tau)^{-\frac{n}{2}}e^{-f}) = -\Delta((4\pi\tau)^{-\frac{n}{2}}e^{-f}) + R((4\pi\tau)^{-\frac{n}{2}}e^{-f}).$$

Thus we can solve the backward heat equation of  $f$  with  $f|_{t=t_0} = f_0$  to obtain  $f(t)$  for  $t \in [0, t_0]$ , satisfying constraint (4.2). Then, for  $t \leq t_0$ , it follows from Lemma 4.2 that

$$\begin{aligned} \mu(g_{ij}(t), T_0 - t) &\leq \mathcal{W}(g_{ij}(t), f(t), T_0 - t) \\ &\leq \mathcal{W}(g_{ij}(t_0), f(t_0), T_0 - t_0) \\ &= \mu(g_{ij}(t_0), T_0 - t_0), \end{aligned}$$

and the second inequality is strict unless we are on a shrinking gradient soliton.  $\square$

#### 4.2 Strong $\kappa$ -noncollapsing of the Ricci flow

We now apply the monotonicity of the  $\mu$ -entropy in Proposition 4.1 to prove a strong version of Perelman's **no local collapsing theorem**, which is extremely important because it gives a local injectivity radius estimate in terms of the local curvature bound.

**Definition 4.1.** Let  $g_{ij}(t)$ ,  $0 \leq t < T$ , be a solution to the Ricci flow on an  $n$ -dimensional manifold  $M$ , and let  $\kappa, r$  be two positive constants. We say that the solution  $g_{ij}(t)$  is  **$\kappa$ -noncollapsed** at  $(x_0, t_0) \in M \times [0, T)$  on the scale  $r$  if we have

$$V_{t_0}(x_0, r) \geq \kappa r^n,$$

whenever

$$|Rm|(x, t_0) \leq r^{-2}$$

for all  $x \in B_{t_0}(x_0, r)$ . Here  $B_{t_0}(x_0, r)$  is the geodesic ball centered at  $x_0 \in M$  and of radius  $r$  with respect to the metric  $g_{ij}(t_0)$ .

*Remark 4.1.* In [55], Perelman also defined  $\kappa$ -noncollapsing by requiring the curvature bound  $|Rm|(x, t) \leq r^{-2}$  on the (backward) parabolic cylinder  $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$ .

The following result was proved in [19] (cf. Theorem 3.3.3 in [19]).

**Theorem 4.1** (Strong no local collapsing theorem). *Let  $M$  be a compact Riemannian manifold, and let  $g_{ij}(t)$  be a solution to the Ricci flow on  $M^n \times [0, T)$  with  $0 < T < +\infty$ . Then there exists a positive constant  $\kappa$ , depending only the initial metric  $g_0$  and  $T$ , such that  $g_{ij}(t)$  is  $\kappa$ -noncollapsed at every point  $(x_0, t_0) \in M \times [0, T)$  on all scales less than  $\sqrt{T}$ . In fact, for any  $(x_0, t_0) \in M \times [0, T)$  and  $0 < r \leq \sqrt{T}$  we have*

$$V_{t_0}(x_0, r) \geq \kappa r^n,$$

whenever

$$R(\cdot, t_0) \leq r^{-2} \quad \text{on } B_{t_0}(x_0, r).$$

**Proof.** Recall that

$$\mu(g_{ij}, \tau) = \inf \left\{ \mathcal{W}(g_{ij}, u, \tau) \mid \int_M (4\pi\tau)^{-\frac{n}{2}} u^2 dV = 1 \right\}.$$

where,

$$\mathcal{W}(g_{ij}, u, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2] dV.$$

Set

$$\mu_0 = \inf_{0 \leq \tau \leq 2T} \mu(g_{ij}(0), \tau) > -\infty. \quad (4.8)$$

By the monotonicity of  $\mu(g_{ij}(t), \tau - t)$  in Proposition 4.1, we have

$$\mu_0 \leq \mu(g_{ij}(0), t_0 + r^2) \leq \mu(g_{ij}(t_0), r^2) \quad (4.9)$$

for  $t_0 < T$  and  $r^2 \leq T$ .

Take a smooth cut-off function  $\zeta(s)$ ,  $0 \leq \zeta \leq 1$ , such that

$$\zeta(s) = \begin{cases} 1, & |s| \leq 1/2, \\ 0, & |s| \geq 1 \end{cases}$$

and  $|\zeta'| \leq 2$  everywhere. Define a test function  $u(x)$  on  $M$  by

$$u(x) = e^{L/2} \zeta\left(\frac{d_{t_0}(x_0, x)}{r}\right),$$

where the constant  $L$  is chosen so that

$$(4\pi r^2)^{-\frac{n}{2}} \int_M u^2 dV_{t_0} = 1$$

Note that

$$|\nabla u|^2 = e^{L} r^{-2} |\zeta'(\frac{d_{t_0}(x_0, x)}{r})|^2 \quad \text{and} \quad u^2 \log u^2 = L u^2 + e^L \zeta^2 \log \zeta^2.$$

Also, by the definition of  $u(x)$ , we have

$$(4\pi r^2)^{-\frac{n}{2}} e^L V_{t_0}(x_0, r/2) \leq 1, \quad (4.10)$$

and

$$(4\pi)^{-\frac{n}{2}} r^{-n} e^L V_{t_0}(x_0, r) \geq 1. \quad (4.11)$$

Now it follows from (4.9) and the upper bound assumption on  $R$  that

$$\begin{aligned} \mu_0 &\leq \mathcal{W}(g_{ij}(t_0), u, r^2) \\ &= (4\pi r^2)^{-\frac{n}{2}} \int_M [r^2(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2] \\ &\leq 1 - L - n + (4\pi r^2)^{-\frac{n}{2}} e^L \int_M (4|\zeta'|^2 - \zeta^2 \log \zeta^2) \\ &\leq 1 - L - n + (4\pi r^2)^{-\frac{n}{2}} e^L (16 + e^{-1}) V_{t_0}(x_0, r). \end{aligned}$$

Here, in the last inequality, we have used the elementary fact that  $-s \log s \leq e^{-1}$  for  $0 \leq s \leq 1$ . Combining the above with (4.10), we arrive at

$$\mu_0 \leq 1 - L - n + (16 + e^{-1}) \frac{V_{t_0}(x_0, r)}{V_{t_0}(x_0, r/2)}. \quad (4.12)$$

Notice that if we have the volume doubling property

$$V_{t_0}(x_0, r) \leq C V_{t_0}(x_0, r/2)$$

for some universal constant  $C > 0$ , then (4.11) and (4.12) together would imply

$$V_{t_0}(x_0, r) \geq \exp\{\mu_0 + n - 1 - (16 + e^{-1})C\}r^n, \quad (4.13)$$

thus proving the theorem. We now describe how to bypass such a volume doubling property by a clever argument<sup>3</sup> pointed out by B.-L. Chen back in 2003.

Notice that the above argument is also valid if we replace  $r$  by any positive number  $0 < a \leq r$ . Thus, at least we have shown the following

**Assertion:** Set

$$\kappa = \min \left\{ \exp[\mu_0 + n - 1 - (16 + e^{-1})3^n], \frac{1}{2}\alpha_n \right\},$$

where  $\alpha_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . Then, for any  $0 < a \leq r$ , we have

$$V_{t_0}(x_0, a) \geq \kappa a^n, \quad (*)_a$$

whenever the volume doubling property,

$$V_{t_0}(x_0, a) \leq 3^n V_{t_0}(x_0, a/2),$$

holds.

Now we finish the proof by contradiction. Suppose  $(*)_a$  fails for  $a = r$ . Then we must have

$$\begin{aligned} V_{t_0}(x_0, \frac{r}{2}) &< 3^{-n} V_{t_0}(x_0, r) \\ &< 3^{-n} \kappa r^n \\ &< \kappa \left(\frac{r}{2}\right)^n. \end{aligned}$$

This says that  $(*)_{r/2}$  would also fail. By induction, we deduce that

$$V_{t_0}(x_0, \frac{r}{2^k}) < \kappa \left(\frac{r}{2^k}\right)^n \quad \text{for all } k \geq 1.$$

But this contradicts the fact that

$$\lim_{k \rightarrow \infty} \frac{V_{t_0}(x_0, \frac{r}{2^k})}{\left(\frac{r}{2^k}\right)^n} = \alpha_n.$$

□

### 4.3 The $\mu$ -entropy and the strong noncollapsing estimate for the NKRF

To convert the  $\kappa$ -noncollapsing theorem for the Ricci flow to the KRF and NKRF, first note that for any local holomorphic coordinates  $(z^1, \dots, z^n)$  with  $z^i = x^i + \sqrt{-1}y^i$ ,  $(x^1, \dots, x^n, y^1, \dots, y^n)$  form a preferred smooth local coordinates with

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial y^i} \right).$$

Thus, in terms of the corresponding Riemannian metric  $ds^2$ , we have

$$ds^2 \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = ds^2 \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 2\Re(g_{i\bar{j}})$$

while

$$ds^2 \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right) = 2\Im(g_{i\bar{j}}).$$

---

<sup>3</sup>Perelman also used a somewhat similar argument in proving his uniform diameter estimate for the NKRF, see the proof of Claim 1 in Section 6.



In particular, for any  $(z^1, \dots, z^n)$  with  $g_{i\bar{j}} = \delta_{ij}$  (e.g., under normal coordinates), then

$$ds^2\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = ds^2\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 2\delta_{ij} \quad \text{and} \quad ds^2\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = 0.$$

(Thus, we can symbolically express the Riemannian metric  $g_{\mathbb{R}} = ds^2 = 2g_{i\bar{j}}$ .)

On the other hand, if  $R_{i\bar{j}} = \lambda\delta_{ij}$  under the normal holomorphic coordinates  $(z^1, \dots, z^n)$  then, for the Riemannian Ricci tensor  $Rc_{ds^2}$ , we have

$$Rc_{ds^2}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = Rc_{ds^2}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 2\lambda\delta_{ij} \quad \text{and} \quad Rc_{ds^2}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = 0.$$

That is,

$$Rc_{ds^2} = \lambda ds^2,$$

so we have the same Einstein constant  $\lambda$ .

Note that we also have the following relations:

- The scalar curvature:  $R_{ds^2} = 2R$
- The Laplace operator:  $\Delta_{ds^2} = 2\Delta$
- The norm square of the gradient of a function:  $|\nabla f|_{ds^2}^2 = 2|\nabla f|^2$ , etc.

In particular, we have

$$R_{ds^2} + |\nabla f|_{ds^2}^2 = 2(R + |\nabla f|^2).$$

Therefore, with  $\sigma = 2\tau$ , the Riemannian  $\mathcal{W}$ -functional on  $(X^n, g_{i\bar{j}})$  is given by

$$\mathcal{W} = \frac{1}{(2\pi\sigma)^n} \int_X [\sigma(R + |\nabla f|^2) + f - 2n] e^{-f} dV, \quad (4.14)$$

or, with  $u = e^{-f/2}$ , by

$$\mathcal{W}(g_{i\bar{j}}, u, \sigma) = \frac{1}{(2\pi\sigma)^n} \int_X [\sigma(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - 2nu^2] dV \quad (4.15)$$

with respect to the Kähler metric  $g_{i\bar{j}}$ .

The  $\mu$ -entropy is then given by

$$\mu = \mu(g_{i\bar{j}}, \sigma) = \inf \left\{ \mathcal{W}(g_{i\bar{j}}, u, \sigma) : (2\pi\sigma)^{-n} \int_X u^2 dV = 1 \right\}.$$

For any solution  $\hat{g}_{i\bar{j}}(s)$  to the KRF on the maximal time interval  $[0, 1)$ , by taking  $\sigma = 1 - s$ , it follows that  $\mu(\hat{g}_{i\bar{j}}(s), 1 - s)$  is monotone increasing in  $s$ . By the scaling invariance property of  $\mu$  in (4.3) and the relation between KRF and NKRF as described in Lemma 2.3, we get

$$\mu(\hat{g}_{i\bar{j}}(s), 1 - s) = \mu(g_{i\bar{j}}(t), 1). \quad (4.16)$$

Thus, by the monotonicity of  $\mu(\hat{g}_{i\bar{j}}(s), 1 - s)$  and  $ds/dt = e^{-t} > 0$ , we have

**Lemma 4.3.** *Let  $g_{i\bar{j}}(t)$  be a solution to the NKRF on  $X^n \times [0, \infty)$ . Then,*

$$\begin{aligned} \mu(g_{i\bar{j}}(t), 1) &= \inf \left\{ \frac{1}{(2\pi)^n} \int_X (R + |\nabla f|^2 + f - 2n) e^{-f} dV : \frac{1}{(2\pi)^n} \int_X e^{-f} dV = 1 \right\} \\ &= \inf \left\{ \frac{1}{(2\pi)^n} \int_X (Ru^2 + 4|\nabla u|^2 - u^2 \log u^2 - 2nu^2) : \frac{1}{(2\pi)^n} \int_X u^2 = 1 \right\} \end{aligned}$$

*is monotone increasing in  $t$ .*

Finally, we have the corresponding strong no local collapsing theorem for the NKRF:

**Theorem 4.2** (Strong no local collapsing theorem for NKRF). *Let  $X^n$  be a Fano manifold, and let  $g_{i\bar{j}}(t)$  be a solution to the NKRF (2.5) on  $X^n \times [0, \infty)$ . Then there exists a positive constant  $\kappa > 0$ , depending only the initial metric  $g_0$ , such that  $g_{i\bar{j}}(t)$  is strongly  $\kappa$ -noncollapsed at very point  $(x_0, t_0) \in M \times [0, \infty)$  on all scales less than  $e^{t_0/2}$  in the following sense: for any  $(x_0, t_0) \in X \times [0, \infty)$  and  $0 < r \leq e^{t_0/2}$  we have*

$$V_{t_0}(x_0, r) \geq \kappa r^{2n}, \quad (4.17)$$

whenever

$$R(\cdot, t_0) \leq r^{-2} \quad \text{on } B_{t_0}(x_0, r). \quad (4.18)$$

**Proof.** This is an immediate consequence of Theorem 4.1 applied to the KRF on  $X^n \times [0, 1)$ , and the relation between the KRF and the NKRF as described by Lemma 2.3.  $\square$

## 5. UNIFORM CURVATURE AND DIAMETER ESTIMATES FOR NKRF WITH NONNEGATIVE BISECTIONAL CURVATURE

Our goal in this section is to prove the uniform diameter and (scalar) curvature estimates by B.L Chen, X.-P. Zhu and the author [14] for the NKRF with nonnegative holomorphic bisectional curvature. The main ingredients of the proof are the Harnack estimate in Theorem 3.4 and the strong non-collapsing estimate in Theorem 4.2 for the NKRF.

**Theorem 5.1.** *Let  $(X^n, \tilde{g}_{i\bar{j}})$  be a compact Kähler manifold with nonnegative bisectional curvature and let  $g_{i\bar{j}}(t)$  be the solution to the NKRF with  $g_{i\bar{j}}(0) = \tilde{g}_{i\bar{j}}$ . Then, there exist positive constants  $C_1 > 0$  and  $C_2 > 0$  such that*

- (i)  $|Rm|(x, t) \leq C_1$  for all  $(x, t) \in X \times [0, \infty)$ ;
- (ii)  $\text{diam}(X^n, g_{i\bar{j}}(t)) \leq C_2$  for all  $t \geq 0$ .

**Proof.** By Theorem 2.2, we know that  $g_{i\bar{j}}(t)$  has nonnegative bisectional curvature for all  $t \geq 0$ . Thus, it suffices to show the uniform upper bound for the scalar curvature

$$R(x, t) \leq C_1$$

on  $X \times [0, \infty)$ . We divide the proof into several steps:

### Step 1: A local uniform bound on $R$

First of all, we know that the volume  $V_t(X^n) = \text{Vol}(X, g_{i\bar{j}}(t))$  and the total scalar curvature  $\int_{X^n} R(x, t) dV_t$  are constant along the NKRF. Hence the average scalar curvature is also constant. In fact,

$$\frac{1}{V_t(X^n)} \int_{X^n} R(x, t) dV_t = n, \quad \text{for all } t \geq 0.$$

Now,  $\forall t > 1$ , set  $t_1 = t$ ,  $t_2 = t + 1$  and pick a point  $y_t \in X$  such that

$$R(y_t, t + 1) = n.$$

Then,  $\forall x \in X$ , by the Harnack inequality in Theorem 3.4, and noting that  $\forall t \geq 1$ ,

$$\frac{e^{t+1} - 1}{e^t - 1} \leq e + 1,$$

we have

$$R(x, t) \leq n(e + 1) \exp\left(\frac{e}{4} d_t^2(x, y_t)\right). \quad (5.1)$$

In particular, when  $d_t(y_t, x) < 1$ , we obtain a uniform upper bound

$$R(\cdot, t) \leq n(e + 1) \exp(e^2/4) \quad (5.2)$$

on the unit geodesic ball  $B_t(y_t, 1)$  at time  $t$ , for all  $t \geq 1$ .

### Step 2: The uniform diameter bound

Now we have the uniform upper bound (5.2) for the scalar curvature on  $B_t(y_t, 1)$ . By applying the strong no local collapsing Theorem 4.2, there exists a positive constant  $\kappa > 0$ , depending only on the initial metric  $g_0$ , such that we have the following uniform lower bound

$$V_t(y_t, 1) \geq \kappa > 0$$

for the volume of the unit geodesic ball  $B_t(y_t, 1)$  for all  $t \geq 1$ .

Suppose  $\text{diam}(X, g_{i\bar{j}}(t))$  is not uniformly bounded from above in  $t$ . Then, there exist a sequence of positive numbers  $\{D_k\} \rightarrow \infty$  and a time sequence  $\{t_k\} \rightarrow \infty$  such that

$$\text{diam}(X, g_{i\bar{j}}(t_k)) > D_k.$$

However, since  $g_{i\bar{j}}(t_k)$  has nonnegative Ricci curvature, it follows from an argument of Yau (cf. p.24 in [65]) that there exists a universal constant  $C = C(n) > 0$  such that

$$V_{t_k}(y_{t_k}, D_k) \geq C V_{t_k}(y_{t_k}, 1) D_k \geq \kappa C D_k \rightarrow \infty.$$

But this contradicts the fact that

$$V_{t_k}(y_{t_k}, D_k) \leq V_{t_k}(X^n) = V_0, \quad k = 1, 2, \dots$$

Thus, we have proved the uniform diameter bound: there exists a positive constant  $D > 0$  such that for all  $t > 0$ ,

$$\text{diam}(X, g_{i\bar{j}}(t)) \leq D. \quad (5.3)$$

### Step 3: The global uniform bound on $R$

Once we have the uniform diameter upper bound (5.3), the Harnack inequality (5.1) immediately implies the uniform scalar curvature upper bound,

$$R(x, t) \leq n(e + 1)e^{eD^2/4},$$

on  $X^n \times [0, \infty)$ . □

*Remark 5.1.* As mentioned in the introduction, assuming in addition the existence of K-E metrics, Chen and Tian studied the NKRF with nonnegative bisectional curvature on Del Pezzo surfaces [23] and Fano manifolds in higher dimensions [24].

## 6. PERELMAN'S UNIFORM SCALAR CURVATURE AND DIAMETER ESTIMATES FOR NKRF

In the previous section, we saw that when a solution  $g_{i\bar{j}}(t)$  to the NKRF has nonnegative bisectional curvature, then the uniform diameter and curvature bounds follow from a nice interplay between the Harnack inequality for the scalar curvature  $R$  and the strong no local collapsing theorem. In this section, we shall see Perelman's amazing uniform estimates on the diameter and the scalar curvature for the NKRF on general Fano manifolds (Theorem 6.1). In absence of the Harnack inequality, Perelman's proof is much more subtle, yet the monotonicity of the  $\mu$ -entropy and the ideas used in the proof of the strong non-collapsing estimate played a crucial role.

The material presented in this section follows closely what Perelman gave in a private lecture at MIT in April, 2003. As such, it naturally overlaps considerably with the earlier notes by Sesum-Tian [67] on Perelman's work<sup>4</sup>. I also presented Perelman's uniform estimates at the Geometry and Analysis seminar at Columbia University in fall 2005.

**Theorem 6.1.** *Let  $X^n$  be a Fano manifold and  $g_{i\bar{j}}(t)$ ,  $0 \leq t < \infty$ , be the solution to the NKRF*

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}, \quad g(0) = \tilde{g} \quad (6.1)$$

*with the initial metric  $g_0 = \tilde{g}$  satisfying  $[\omega_0] = \pi c_1(X)$ . Let  $f = f(t)$  be the Ricci potential of  $g_{i\bar{j}}(t)$  satisfying*

$$-R_{i\bar{j}}(t) + g_{i\bar{j}}(t) = \partial_i \partial_{\bar{j}} f \quad (6.2)$$

*and the normalization*

$$\int_{X^n} e^{-f} dV = (2\pi)^n. \quad (6.3)$$

*Then there exists a constant  $C > 0$  such that*

- (i)  $|R| \leq C$  on  $X^n \times [0, \infty)$ ;
- (ii)  $\text{diam}(X^n, g_{i\bar{j}}(t)) \leq C$ ;
- (iii)  $\|f\|_{C^1} \leq C$  on  $X^n \times [0, \infty)$ .

**Proof.** First of all, by Lemma 2.5, we know that under (6.1) the scalar curvature  $R$  evolves according to the equation

$$\frac{\partial}{\partial t} R = \Delta R + |Rc|^2 - R.$$

**Lemma 6.1.** *There exists a constant  $C_1 > 0$  such that the scalar curvature  $R$  of the NKRF (6.1) satisfies the estimate*

$$R(x, t) \geq -C_1.$$

*for all  $t \geq 0$  and all  $x \in X^n$ .*

---

<sup>4</sup>Perelman's private lecture was attended by a very small audience, including this author and the authors of [67].

**Proof.** Let  $R_{\min}(0)$  be the minimum of  $R(x, 0)$  on  $X^n$ . If  $R_{\min}(0) \geq 0$ , then by the maximum principle, we have  $R(x, t) \geq 0$  for all  $t > 0$  and all  $x \in X^n$ .

Now suppose  $R_{\min}(0) < 0$ . Set  $F(x, t) = R(x, t) - R_{\min}(0)$ . Then,  $F(x, 0) \geq 0$  and  $F$  satisfies

$$\frac{\partial}{\partial t} F = \Delta F + |Rc|^2 - F - R_{\min}(0) > \Delta F + |Rc|^2 - F.$$

Hence it follows again from the maximum principle that  $F \geq 0$  on  $X^n \times [0, \infty)$ , i.e.,

$$R(x, t) \geq R_{\min}(0)$$

for all  $t > 0$  and all  $x \in X^n$ .  $\square$

Next, we consider the Ricci potential  $f$  satisfying (6.2) and the normalization (6.3). Note that it follows from (6.2) that

$$n - R = \Delta f. \quad (6.4)$$

Also, let  $\varphi = \varphi(t)$  be the Kähler potential,

$$g_{i\bar{j}}(t) = \tilde{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi,$$

so that  $\varphi$  is a solution to the parabolic scalar equation

$$\varphi_t = \log \frac{\det(\tilde{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)}{\det(\tilde{g}_{i\bar{j}})} + \tilde{f} + \varphi + b(t),$$

where  $b(t)$  is a function of  $t$  only.

Since  $\partial_i \partial_{\bar{j}} \varphi_t = -R_{i\bar{j}} + g_{i\bar{j}}$ , by adding a function of  $t$  only to  $\varphi$  if necessary, we can assume

$$f = \varphi_t. \quad (6.5)$$

Thus,  $f$  satisfies the parabolic equation

$$f_t = \Delta f + f - a(t) \quad (6.6)$$

for some function  $a(t)$  of  $t$  only.

By differentiating the constraint (6.3), we get

$$\int_{X^n} e^{-f} (-f_t + n - R) dV = 0.$$

Hence, by combining with (6.4) and (6.6), it follows that

$$a(t) = (2\pi)^{-n} \int_{X^n} f e^{-f} dV. \quad (6.7)$$

**Lemma 6.2.** *There exists a constant  $C_2 > 0$  such that, for all  $t \geq 0$ ,*

$$-C_2 \leq \int_{X^n} f e^{-f} dV \leq C_2.$$

**Proof.** The second inequality is easy to see. Now we prove the first inequality. By Lemma 4.3 and (6.4), we have

$$\begin{aligned} A &= \mu(g_{i\bar{j}}(0), 1) \leq \mu(g_{i\bar{j}}(t), 1) \\ &\leq (2\pi)^{-n} \int_X (R + |\nabla f|^2 + f - 2n)e^{-f} dV \\ &= (2\pi)^{-n} \int_X (-\Delta f + |\nabla f|^2 + f - n)e^{-f} dV \\ &= (2\pi)^{-n} \int_X (f - n)e^{-f} dV. \end{aligned}$$

Therefore,

$$(2\pi)^{-n} \int_{X^n} f e^{-f} dV \geq A + n.$$

□

**Lemma 6.3.** *There exists a constant  $C_3 > 0$  such that*

$$f \geq -C_3$$

for all  $t \geq 0$  and all  $x \in X^n$ .

**Proof.** We argue by contradiction. Suppose the Ricci potential  $f$  is very negative at some time  $t_0 > 0$  and some point  $x_0 \in X^n$  so that

$$f(x_0, t_0) \ll -1.$$

Then, there exists some open neighborhood  $U \subset X^n$  of  $x_0$  such that

$$f(x, t_0) \ll -1, \quad \forall x \in U. \quad (6.8)$$

On the other hand, by (6.4), (6.6), Lemma 6.1, (6.7), and Lemma 6.2, we have

$$f_t = n - R + f - a(t) \leq f + C \quad (6.9)$$

for some uniform constant  $C > 0$ .

Let us assume  $f(\cdot, t)$  and  $\varphi(\cdot, t)$  achieve their maximum at  $x_t$  and  $x_t^*$  respectively. From the constraint (6.3), it is clear that for each  $t > 0$ , we have a uniform lower estimate

$$f(x_t, t) = \max_X f(\cdot, t) \geq -C$$

for some  $C > 0$  independent of  $t$ . Moreover, it follows from (6.5) and (6.9) that

$$(f - \varphi)_t \leq C,$$

so

$$f(\cdot, t) - \varphi(\cdot, t) \leq \max_X (f - \varphi)(\cdot, t_0) + Ct.$$

Therefore,

$$\varphi(x_t^*, t) \geq \varphi(x_t, t) \geq f(x_t, t) - \max_X (f - \varphi)(\cdot, t_0) - Ct \geq -Ct, \quad \forall t \gg t_0. \quad (6.10)$$

On the other other, by (6.9), we have

$$f(x, t) \leq e^{t-t_0} (C + f(x, t_0)) \quad (6.11)$$

for  $t \geq t_0$  and  $x \in X^n$ . In particular, by (6.8), we have

$$f(x, t) \leq -Ce^{-t_0} e^t, \quad \forall t > t_0, \forall x \in U. \quad (6.12)$$

Then (6.5) and (6.12) together imply that

$$\varphi(x, t) \leq \varphi(x, t_0) - Ce^{-t_0}e^t + C \leq -C'e^t, \quad \forall t \gg t_0, \forall x \in U. \quad (6.13)$$

Next, we claim (6.13) implies

$$\varphi(x_t^*, t) \leq -Ce^t + C' \quad (6.14)$$

for some  $C' > 0$  independent of  $t \gg t_0$ . To see this, note that, with respect to the initial metric  $g_0$ , we have

$$\varphi(x_t^*, t) = \frac{1}{V_0(X^n)} \int_X \varphi(\cdot, t) dV_0 - \frac{1}{V_0(X^n)} \int_X \Delta_0 \varphi(\cdot, t) G_0(x_t^*, \cdot) dV_0, \quad (6.15)$$

where  $V_0(X^n) = \text{Vol}(X^n, g_0)$  and  $G_0(x_t^*, \cdot)$  denotes a positive Green's function with pole at  $x_t^*$ .

Since  $n + \Delta_0 \varphi = \tilde{g}^{i\bar{j}} g_{i\bar{j}}(t) > 0$ , the second term on the RHS of (6.15) can be estimated by

$$-\frac{1}{V_0(X^n)} \int_X \Delta_0 \varphi(\cdot, t) G_0(x_t, \cdot) dV_0 \leq \frac{n}{V_0(X^n)} \int_X G_0(x_t, \cdot) dV_0 =: C''. \quad (6.16)$$

On the other hand, by using (6.12), it follows that

$$\frac{1}{V_0(X^n)} \int_X \varphi(\cdot, t) dV_0 \leq \frac{V_0(X \setminus U)}{V_0(X)} \varphi(x_t^*, t) - \frac{V_0(U)}{V_0(X)} Ce^t. \quad (6.17)$$

Therefore, by (6.15)-(6.17), we have

$$\alpha \varphi(x_t^*, t) \leq C'' - \alpha Ce^t$$

for  $\alpha = V_0(U)/V_0(X) > 0$ . This proves (6.14), a contradiction to (6.10).  $\square$

**Lemma 6.4.** *There exists constant  $C_4 > 0$  such that, for all  $t \geq 0$ ,*

- (a)  $|\nabla f|^2 \leq C_4(f + 2C_3)$ ;
- (b)  $R \leq C_4(f + 2C_3)$ .

**Proof.** This is essentially a parabolic version of Yau's gradient estimate in [83] (see also [65]).

First of all, from  $|\nabla f|^2 = g^{i\bar{j}} \partial_i f \partial_{\bar{j}} f$ , the NKRF, and (6.6), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla f|^2 &= (R_{i\bar{j}} - g_{i\bar{j}}) \partial_i f \partial_{\bar{j}} f + g^{i\bar{j}} (\partial_i f_t \partial_{\bar{j}} f + \partial_i f \partial_{\bar{j}} f_t) \\ &= g^{i\bar{j}} [\partial_i (\Delta f) \partial_{\bar{j}} f + \partial_i f \partial_{\bar{j}} (\Delta f)] + Rc(\nabla f, \nabla f) + |\nabla f|^2. \end{aligned}$$

On the other hand, the Bochner formula gives us

$$\Delta |\nabla f|^2 = |\nabla \bar{\nabla} f|^2 + |\nabla \nabla f|^2 + g^{i\bar{j}} [\partial_i (\Delta f) \partial_{\bar{j}} f + \partial_i f \partial_{\bar{j}} (\Delta f)] + Rc(\nabla f, \nabla f).$$

Hence, we have

$$\frac{\partial}{\partial t} |\nabla f|^2 = \Delta |\nabla f|^2 - |\nabla \bar{\nabla} f|^2 - |\nabla \nabla f|^2 + |\nabla f|^2. \quad (6.18)$$

Also, by (6.2), we have

$$|Rc|^2 + n - 2R = |\nabla \bar{\nabla} f|^2. \quad (6.19)$$

Thus, from the evolution equation on  $R$ , we have

$$\frac{\partial}{\partial t} R \leq \Delta R + |\nabla \bar{\nabla} f|^2 + R$$

Therefore, for any  $\alpha \geq 0$ , we obtain

$$\frac{\partial}{\partial t}(|\nabla f|^2 + \alpha R) \leq \Delta(|\nabla f|^2 + \alpha R) - (1 - \alpha)(|\nabla \bar{\nabla} f|^2 + |\nabla \nabla f|^2) + (|\nabla f|^2 + \alpha R). \quad (6.20)$$

Next, take  $B = 2C_3$  so we have  $f + B > 1$ , and set

$$u = \frac{|\nabla f|^2 + \alpha R}{f + B}. \quad (6.21)$$

Then, we have

$$u_t = \frac{(|\nabla f|^2 + \alpha R)_t}{f + B} - \frac{u}{(f + B)} f_t$$

and

$$\nabla u = \frac{1}{f + B} \nabla(|\nabla f|^2 + \alpha R) - \frac{|\nabla f|^2 + \alpha R}{(f + B)^2} \nabla f. \quad (6.22)$$

On the other hand, since  $|\nabla f|^2 + \alpha R = u(f + B)$ , we have

$$\Delta(|\nabla f|^2 + \alpha R) = (f + B)\Delta u + u\Delta f + \nabla u \cdot \bar{\nabla} f + \bar{\nabla} u \cdot \nabla f$$

or

$$\Delta u = \frac{\Delta(|\nabla f|^2 + \alpha R)}{f + B} - \frac{u\Delta f}{f + B} - \frac{\nabla u \cdot \bar{\nabla} f + \bar{\nabla} u \cdot \nabla f}{f + B}.$$

Therefore,

$$u_t \leq \Delta u - (1 - \alpha) \frac{(|\nabla \bar{\nabla} f|^2 + |\nabla \nabla f|^2)}{f + B} + \frac{\nabla u \cdot \bar{\nabla} f + \bar{\nabla} u \cdot \nabla f}{f + B} + \frac{B + a(t)}{f + B} u. \quad (6.23)$$

Notice, by (6.22), we have

$$\nabla u \cdot \bar{\nabla} f = \frac{1}{f + B} \nabla(|\nabla f|^2 + \alpha R) \cdot \bar{\nabla} f - \frac{(|\nabla f|^2 + \alpha R)|\nabla f|^2}{(f + B)^2}. \quad (6.24)$$

Now the trick (see, e.g., p. 19 in [65]) is to use (6.24) and express

$$\frac{\nabla u \cdot \bar{\nabla} f}{f + B} = (1 - 2\epsilon) \frac{\nabla u \cdot \bar{\nabla} f}{f + B} + \frac{2\epsilon}{f + B} \left( \frac{\nabla(|\nabla f|^2 + \alpha R) \cdot \bar{\nabla} f}{f + B} - \frac{|\nabla f|^2(|\nabla f|^2 + \alpha R)}{(f + B)^2} \right). \quad (6.25)$$

We are ready to conclude the proof of Lemma 6.4.

**Part (a):** Take  $\alpha = 0$  so that  $u = |\nabla f|^2/(f + B)$ . By plugging (6.25) into (6.23), we get

$$\begin{aligned} u_t &\leq \Delta u - (1 - 4\epsilon) \frac{|\nabla \bar{\nabla} f|^2 + |\nabla \nabla f|^2}{f + B} + (1 - 2\epsilon) \frac{\nabla u \cdot \bar{\nabla} f + \bar{\nabla} u \cdot \nabla f}{f + B} \\ &\quad - \frac{\epsilon}{f + B} \left( \left| 2\nabla \bar{\nabla} f - \frac{\nabla f \bar{\nabla} f}{f + B} \right|^2 + \left| 2\nabla \nabla f - \frac{\nabla f \nabla f}{f + B} \right|^2 \right) \\ &\quad + \frac{1}{(f + B)} (-2\epsilon u^2 + (B + a)u). \end{aligned}$$

For any  $T > 0$ , suppose  $u$  attains its maximum at  $(x_0, t_0)$  on  $X^n \times [0, T]$ , then

$$u_t(x_0, t_0) \geq 0, \quad \nabla u(x_0, t_0) = 0, \quad \text{and} \quad \Delta u(x_0, t_0) \leq 0. \quad (6.26)$$

Thus, by choosing  $\epsilon = 1/8$ , we arrive at

$$u(x_0, t_0) \leq 4(B + a).$$



Therefore, since  $T > 0$  is arbitrary, we have shown that

$$\frac{|\nabla f|^2}{f+B} \leq 8C_3 + 4C_2 \quad (6.27)$$

on  $X^n \times [0, \infty)$ .

**Part (b):** Choose  $\alpha = 1/2$  so that

$$u = \frac{|\nabla f|^2 + R/2}{f+B}.$$

Then, from (6.23) and (6.19), we obtain

$$u_t \leq \Delta u - \frac{1}{2} \frac{|Rc|^2 - 2R}{f+B} + \frac{\nabla u \cdot \bar{\nabla} f + \bar{\nabla} u \cdot \nabla f}{f+B} + \frac{B+a}{f+B} u.$$

Again, for any  $T > 0$ , suppose  $u$  attains its maximum at  $(x_0, t_0)$  on  $X^n \times [0, T]$ . Then (6.26) holds, and hence

$$0 \leq -\frac{1}{2n} \left( \frac{R}{f+B} \right)^2 (x_0, t_0) + \frac{R}{f+B} (x_0, t_0) \left( 1 + \frac{B+a}{2(f+B)} \right) + (8C_3 + 4C_2)(B+a).$$

Here we have used the fact that  $|Rc|^2 \geq R^2/n$ ,  $2f+B \geq 0$ ,  $f+B \geq 1$ , and (6.27). It then follows easily that  $\frac{R}{f+B}(x_0, t_0)$  is bounded from above uniformly. Therefore, by Part (a),  $\frac{R}{f+B}(x, t)$  is bounded uniformly on  $X^n \times [0, T]$  for arbitrary  $T > 0$ .  $\square$

Clearly, Lemma 6.4 (a) implies that  $\sqrt{f+2C_3}$  is Lipschitz. From now on we assume the Ricci potential  $f(\cdot, t)$  attains its minimum at a point  $\hat{x} \in X^n$ , i.e.,  $f(\hat{x}, t) = \min_X f(\cdot, t)$ . Then, by (6.3), we know

$$f(\hat{x}, t) \leq C$$

for some  $C > 0$  independent of  $t$ .

**Corollary 6.1.** *There exists a constant  $C > 0$  such that  $\forall t > 0$  and  $\forall x \in X$ ,*

- (i)  $f(x, t) \leq C[1 + d_t^2(\hat{x}, x)]$ ;
- (ii)  $|\nabla f|^2(x, t) \leq C[1 + d_t^2(\hat{x}, x)]$ ;
- (iii)  $R(x, t) \leq C[1 + d_t^2(\hat{x}, x)]$ .

**Proof.** Set  $h = f + 2C_2 > 0$ . Then, from Lemma 6.4 (i), we see that  $\sqrt{h}$  is a Lipschitz function satisfying

$$|\nabla \sqrt{h}|^2 \leq C_4.$$

Hence,  $\forall x \in X^n$ ,

$$|\sqrt{h}(x, t) - \sqrt{h}(\hat{x}, t)| \leq C d_t(\hat{x}, x),$$

or

$$\sqrt{h}(x, t) \leq \sqrt{h}(\hat{x}, t) + C d_t(\hat{x}, x).$$

Thus, we obtain a uniform upper bound

$$f(x, t) \leq h(x, t) \leq C(d_t^2(\hat{x}, y) + 1)$$

for some constant  $C > 0$  independent of  $t$ . Now (ii) and (iii) follow immediately from (i) and Lemma 6.4.  $\square$

By Lemma 6.1 and Corollary 6.5, it remains to prove the following uniform diameter bound.

**Lemma 6.5.** *There exists a constant  $C_5 > 0$  such that*

$$\text{diam}_t(X) =: \text{diam}(X^n, g_{i\bar{j}}(t)) \leq C_5$$

for all  $t \geq 0$ .

*Proof.* For each  $t > 0$ , denote by  $A_t(k_1, k_2)$  the annulus region defined by

$$A_t(k_1, k_2) = \{z \in X : 2^{k_1} \leq d_t(x, \hat{x}) \leq 2^{k_2}\}, \quad (6.28)$$

and by

$$V_t(k_1, k_2) = \text{Vol}(A_t(k_1, k_2)) \quad (6.29)$$

with respect to  $g_{i\bar{j}}(t)$ .

Note that each annulus  $A_t(k, k+1)$  contains at least  $2^{2k}$  balls  $B_r$  of radius  $r = 2^{-k}$ . Also, for each point  $x \in A_t(k, k+1)$ , Corollary 6.1 (iii) implies that the scalar curvature is bounded above by  $R \leq C2^{2k}$  on  $B_t(x, r)$  for some uniform constant  $C > 0$ . Thus each of these balls  $B_r$  has  $\text{Vol}(B_r) \geq \kappa(2^{-k})^{2n}$  by Theorem 4.2, so we have

$$V_t(k, k+1) \geq \kappa 2^{2k-1} 2^{-kn}. \quad (6.30)$$

**Claim 6.1:** For each small  $\epsilon > 0$ , there exists a large constant  $D = D(\epsilon) > 0$  such that if  $\text{diam}_t(X) > D$ , then one can find large positive constants  $k_2 > k_1 > 0$  with the following properties:

$$V_t(k_1, k_2) \leq \epsilon \quad (6.31)$$

and

$$V_t(k_1, k_2) \leq 2^{10n} V_t(k_1 + 2, k_2 - 2). \quad (6.32)$$

*Proof.* (a) follows from the fact that  $V_t(X^n) = V_0(X^n)$  and the assumption  $\text{diam}_t(X) \gg 1$ .

Now suppose (a) holds but not (b), i.e.,

$$V_t(k_1, k_2) > 2^{10n} V_t(k_1 + 2, k_2 - 2).$$

Then we consider whether or not

$$V_t(k_1 + 2, k_2 - 2) \leq 2^{10n} V_t(k_1 + 4, k_2 - 4).$$

If yes, then we are done. Otherwise we repeat the process.

After  $j$  steps, we either have

$$V_t(k_1 + 2(j-1), k_2 - 2(j-1)) \leq 2^{10nj} V_t(k_1 + 2j, k_2 - 2j), \quad (6.33)$$

or

$$V_t(k_1, k_2) > 2^{10nj} V_t(k_1 + 2j, k_2 - 2j). \quad (6.34)$$

Without loss of generality, we may assume  $k_1 + 2j \approx k_2 - 2j$  by choosing a large number  $K > 0$  and pick  $k_1 \approx K/2, k_2 \approx 3K/2$ . Then, when  $j \approx K/4$  and using (6.30), this implies that

$$\epsilon \geq V_t(k_1, k_2) \geq 2^{10nK/4} V_t(K, K+1) \geq \kappa 2^{2K(n/4-1)}.$$

So after some finitely many steps  $j \approx K(\epsilon)/4$ , (6.33) must hold. Therefore, we have found  $k_1$  and  $k_2 \approx 3k_1$  satisfying both (6.31) and (6.32).

**Claim 6.2:** There exist constants  $r_1 > 0$  and  $r_2 > 0$ , with  $r_1 \in [2^{k_1}, 2^{k_1+1}]$  and  $r_2 \in [2^{k_2}, 2^{k_2+1}]$ , such that

$$\int_{A_t(r_1, r_2)} R dV_t \leq C V_t(k_1, k_2). \quad (6.35)$$

*Proof.* First of all, since

$$\frac{d}{dr} \text{Vol}(B(r)) = \text{Vol}(S(r)),$$

we have

$$V(k_1, k_1 + 1) = \int_{2^{k_1}}^{2^{k_1+1}} \text{Vol}(S(r)) dr.$$

Here  $S_r$  denotes the geodesic sphere of radius  $r$  centered at  $\hat{x}$  with respect to  $g_{i\bar{j}}(t)$ . Hence, we can choose  $r_1 \in [2^{k_1}, 2^{k_1+1}]$  such that

$$\text{Vol}(S_{r_1}) \leq \frac{V_t(k_1, k_2)}{2^{k_1}},$$

for otherwise

$$V(k_1, k_1 + 1) > \frac{V_t(k_1, k_2)}{2^{k_1}} 2^{k_1} = V_t(k_1, k_2),$$

a contradiction because  $k_2 > k_1 + 1$ . Similarly, there exists  $r_2 \in [2^{k_2-1}, 2^{k_2}]$  such that

$$\text{Vol}(S_{r_2}) \leq \frac{V_t(k_1, k_2)}{2^{k_2}}.$$

Next, by integration by parts and Corollary 6.1(ii),

$$\begin{aligned} \left| \int_{A_t(r_1, r_2)} \Delta f \right| &\leq \int_{S_{r_1}} |\nabla f| + \int_{S_{r_2}} |\nabla f| \\ &\leq \frac{V_t(k_1, k_2)}{2^{k_1}} C 2^{k_1+1} + \frac{V_t(k_1, k_2)}{2^{k_2}} C 2^{k_2+1} \\ &\leq C V_t(k_1, k_2). \end{aligned}$$

Therefore, since  $R + \Delta f = n$ , it follows that

$$\int_{A_t(r_1, r_2)} R dV_t \leq C V_t(k_1, k_2),$$

proving Claim 6.2.

Now we argue by contradiction to finish the proof: Suppose  $\text{diam}_t(X^n)$  is unbounded for  $0 \leq t < \infty$ . Then, for any sequence  $\epsilon_i \rightarrow 0$ , there exists a time sequence  $\{t_i\} \rightarrow \infty$  and  $k_2^{(i)} > k_1^{(i)} > 0$  for which Claim 6.1 holds. Pick smooth cut-off functions  $0 \leq \zeta_i(s) \leq 1$  defined on  $\mathbb{R}$  such that

$$\zeta_i(s) = \begin{cases} 1, & 2^{k_1^{(i)}+2} \leq s \leq 2^{k_2^{(i)}-2}, \\ 0, & \text{outside } [r_1^{(i)}, r_2^{(i)}], \end{cases}$$

and  $|\zeta'| \leq 1$  everywhere. Here  $r_1^{(i)} \in [2^{k_1^{(i)}}, 2^{k_1^{(i)}+1}]$  and  $r_2^{(i)} \in [2^{k_2^{(i)}-1}, 2^{k_2^{(i)}}]$  are chosen as in Claim 6.2. Define

$$u_i = e^{L_i} \zeta_i(d_{t_i}(x, \hat{x}_i)),$$

where  $f(\hat{x}_i, t_i) = \min_X f(\cdot, t_i)$  and the constant  $L_i$  is chosen so that

$$(2\pi)^n = \int_X u_i^2 dV_{t_i} = e^{2L_i} \int_{A(r_1^{(i)}, r_2^{(i)})} \zeta_i^2 dV_{t_i}. \quad (6.36)$$

Note that by Claim 1,  $V_{t_i}(k_1^{(i)}, k_2^{(i)}) \leq \epsilon_i \rightarrow 0$ . Hence (6.36) implies  $L_i \rightarrow \infty$ .

Now, by Lemma 4.3 and similar to the proof of Theorem 4.1, we have

$$\begin{aligned} \mu(g(0), 1) &\leq \mu(g(t_i), 1) \\ &\leq (2\pi)^{-n} \int_X (Ru_i^2 + 4|\nabla u_i|^2 - u_i^2 \log u_i^2 - 2nu_i^2) dV_{t_i} \\ &= (2\pi)^{-n} e^{2L_i} \int_{A_{t_i}(r_1^{(i)}, r_2^{(i)})} (R\zeta_i^2 + 4|\zeta_i'|^2 - \zeta_i^2 \log \zeta_i^2 - 2L_i \zeta_i^2 - 2n\zeta_i^2) dV_{t_i} \\ &= -2(L_i + n) + (2\pi)^{-n} e^{2L_i} \int_{A_{t_i}(r_1^{(i)}, r_2^{(i)})} (R\zeta_i^2 + 4|\zeta_i'|^2 - \zeta_i^2 \log \zeta_i^2) dV_{t_i}. \end{aligned}$$

Now, by Claim 6.2 and Claim 6.1, we have

$$\begin{aligned} e^{2L_i} \int_{A_{t_i}(r_1^{(i)}, r_2^{(i)})} R\zeta_i^2 dV_{t_i} &\leq C e^{2L_i} V_{t_i}(k_1^{(i)}, k_2^{(i)}) \\ &\leq C e^{2L_i} 2^{10n} V_{t_i}(k_1^{(i)} + 2, k_2^{(i)} - 2) \\ &\leq C 2^{10n} \int_{A_{t_i}(r_1^{(i)}, r_2^{(i)})} u_i^2 dV_{t_i} \leq C 2^{10n} (2\pi)^n. \end{aligned}$$

On the other hand, using  $|\zeta_i'| \leq 1$  and  $-s \log s \leq e^{-1}$  for  $0 \leq s \leq 1$ , we also have

$$\begin{aligned} e^{2L_i} \int_{A_{t_i}(r_1^{(i)}, r_2^{(i)})} (4|\zeta_i'|^2 - 2\zeta_i^2 \log \zeta_i) dV_{t_i} &\leq C e^{2L_i} V_{t_i}(k_1^{(i)}, k_2^{(i)}) \\ &\leq C 2^{10n} (2\pi)^n. \end{aligned}$$

Therefore,

$$\mu(g(0), 1) \leq -2(L_i + n) + C$$

for some uniform constant  $C > 0$ . But this is a contradiction to  $\{L_i\} \rightarrow \infty$ .  $\square$

## 7. REMARKS ON THE FORMATION OF SINGULARITIES IN KRF

Consider a solution  $g_{ij}(t)$  to the Ricci flow

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(t)$$

on  $M \times [0, T)$ ,  $T \leq +\infty$ , where either  $M$  is compact or at each time  $t$  the metric  $g_{ij}(t)$  is complete and has bounded curvature. We say that  $g_{ij}(t)$  is a *maximal* solution of the Ricci flow if either  $T = +\infty$  or  $T < +\infty$  and the norm of its curvature tensor  $|Rm|$  is unbounded as  $t \rightarrow T$ . In the latter case, we say  $g_{ij}(t)$  is a *singular* solution to the Ricci flow with singular time  $T$ . We emphasize that by singular solution  $g_{ij}(t)$  we mean the curvature of  $g_{ij}(t)$  is not uniformly bounded on  $M^n \times [0, T)$ , while  $M^n$  is a smooth manifold and  $g_{ij}(t)$  is a smooth complete metric for each  $t < T$ .

As in the minimal surface theory and harmonic map theory, one usually tries to understand the structure of a singularity by rescaling the solution (or blow up) to

obtain a sequence of solutions and study its limit. For the Ricci flow, the theory was first developed by Hamilton in [42] and further improved by Perelman [55, 56].

Now we apply Hamilton's theory to investigate singularity formations of KRF (2.1) on compact Fano manifolds. Consider a (maximal) solution  $\hat{g}_{i\bar{j}}(s)$  to KRF (2.1) on  $X^n \times [0, 1)$  and the corresponding solution  $g_{i\bar{j}}(t)$  to NKRF (2.5) on  $X^n \times [0, \infty)$ , and let us denote by

$$\hat{K}_{\max}(s) = \max_X |\hat{Rm}(\cdot, s)|_{\hat{g}(s)} \quad \text{and} \quad K_{\max}(t) = \max_X |Rm(\cdot, t)|_{g(t)}.$$

According to Hamilton [42], one can classify maximal solutions to KRF (2.1) on any compact Fano manifold  $X^n$  into Type I and Type II:

**Type I:**  $\limsup_{s \rightarrow 1} (1 - s) \hat{K}_{\max}(s) < +\infty;$

**Type II:**  $\limsup_{s \rightarrow 1} (1 - s) \hat{K}_{\max}(s) = +\infty.$

On the other hand, by Corollary 2.1,  $\hat{K}_{\max}(s)$  and  $K_{\max}(t)$  are related by

$$(1 - s) \hat{K}_{\max}(s) = K_{\max}(t(s)).$$

Thus, we immediately get

**Lemma 7.1.** *Let  $\hat{g}_{i\bar{j}}(s)$  be a solution to KRF (2.1) on  $X^n \times [0, 1)$  and  $g_{i\bar{j}}(t)$  be the corresponding solution to NKRF (2.5) on  $X^n \times [0, \infty)$ . Then,*

- (a)  $\hat{g}_{i\bar{j}}(s)$  is a Type I solution if and only if  $g_{i\bar{j}}(t)$  is a nonsingular solution, i.e.,  $K_{\max}(t) \leq C$  for some constant  $C > 0$  for all  $t \in [0, \infty)$ ;
- (b)  $\hat{g}_{i\bar{j}}(s)$  is a Type II solution if and only if  $g_{i\bar{j}}(t)$  is a singular solution.

For each type of (maximal) solutions  $\hat{g}_{i\bar{j}}(s)$  to KRF (2.1) or the corresponding solutions  $g_{i\bar{j}}(t)$  for NKRF (2.5), following Hamilton [42] (see also Chapter 4 of [19]) we define a corresponding type of limiting singularity models.

**Definition 7.1.** A solution  $g_{i\bar{j}}^\infty(t)$  to KRF on a complex manifold  $X_\infty^n$  with complex structure  $J_\infty$ , where either  $X_\infty^n$  is compact or at each time  $t$  the Kähler metric  $g_{i\bar{j}}^\infty(t)$  is complete and has bounded curvature, is called a Type I or Type II **singularity model** if it is not flat and of one of the following two types:

**Type I:**  $g_{i\bar{j}}^\infty(t)$  exists for  $t \in (-\infty, \Omega)$  for some  $\Omega$  with  $0 < \Omega < +\infty$  and

$$|Rm^\infty|(x, t) \leq \Omega/(\Omega - t)$$

everywhere on  $X_\infty^n \times (-\infty, \Omega)$  with equality somewhere at  $t = 0$ ;

**Type II:**  $g_{i\bar{j}}^\infty(t)$  exists for  $t \in (-\infty, +\infty)$  and

$$|Rm^\infty|(x, t) \leq 1$$

everywhere on  $X_\infty^n \times (-\infty, \Omega)$  with equality somewhere at  $t = 0$ .

With the help of the strong  $\kappa$ -noncollapsing theorem, we can apply Hamilton's Type I and Type II blow up arguments to get the following result, a Kähler analog of Theorem 16.2 in [42]:

**Theorem 7.1.** *For any (maximal) solution  $\hat{g}_{i\bar{j}}(s)$ ,  $0 \leq s < 1$ , to KRF (2.1) on compact Fano manifold  $X^n$  (or the corresponding solution  $g_{i\bar{j}}(t)$  to NKRF (2.5) on  $X^n \times [0, \infty)$ ), which is of either Type I or Type II, there exists a sequence of dilations of the solution which converges in  $C_{loc}^\infty$  topology to a singularity model  $(X_\infty^n, J_\infty, g^\infty(t))$  of the corresponding Type. Moreover, the Type I singularity model*

$(X_\infty^n, J_\infty, g^\infty(t))$  is compact with  $X_\infty^n = X^n$  as a smooth manifold, while the Type II singularity model  $(X_\infty^n, J_\infty, g^\infty(t))$  is complete noncompact.

**Proof. Type I case:** Let

$$\Omega =: \limsup_{t \rightarrow 1} (1-s) \hat{K}_{\max}(s) < +\infty.$$

First we note that  $\Omega > 0$ . Indeed by the evolution equation of curvature,

$$\frac{d}{ds} \hat{K}_{\max}(s) \leq \text{Const} \cdot \hat{K}_{\max}^2(s).$$

This implies that

$$\hat{K}_{\max}(s) \cdot (1-s) \geq \text{Const} > 0,$$

because

$$\limsup_{t \rightarrow 1} \hat{K}_{\max}(s) = +\infty.$$

Thus  $\Omega$  must be positive.

Next we choose a sequence of points  $x_k$  and times  $s_k$  such that  $s_k \rightarrow 1$  and

$$\lim_{k \rightarrow \infty} (1-s_k) |\hat{R}m|(x_k, s_k) = \Omega.$$

Denote by

$$Q_k = |\hat{R}m|(x_k, s_k).$$

Now translate the time so that  $s_k$  becomes 0 in the new time, and dilate in space-time by the factor  $Q_k$  (time like distance squared) to get the rescaled solution

$$\hat{g}_{i\bar{j}}^{(k)}(\hat{t}) = Q_k \hat{g}_{i\bar{j}}(s_k + Q_k^{-1}\hat{t})$$

to the KRF

$$\frac{\partial}{\partial \hat{t}} \hat{g}_{i\bar{j}}^k = -2\hat{R}_{i\bar{j}}^{(k)},$$

where  $\hat{R}_{i\bar{j}}^{(k)}$  is the Ricci tensor of  $\hat{g}_{i\bar{j}}^{(k)}$ , on the time interval  $[-Q_k s_k, Q_k(1-s_k))$ , with

$$Q_k s_k = s_k |\hat{R}m|(x_k, s_k) \rightarrow \infty \quad \text{and} \quad Q_k(1-s_k) = (1-s_k) |\hat{R}m|(x_k, s_k) \rightarrow \Omega.$$

For any  $\epsilon > 0$  we can find a time  $\tau < 1$  such that for  $s \in [\tau, 1)$ ,

$$|\hat{R}m| \leq (\Omega + \epsilon)/(1-s)$$

by the assumption. Then for  $\hat{t} \in [Q_k(\tau - s_k), Q_k(1-s_k))$ , the curvature of  $\hat{g}_{i\bar{j}}^{(k)}(\hat{t})$  is bounded by

$$\begin{aligned} |\hat{R}m^{(k)}| &= Q_k^{-1} |\hat{R}m(\hat{g})| \\ &\leq \frac{\Omega + \epsilon}{Q_k(1-s)} = \frac{\Omega + \epsilon}{Q_k(1-s_k) + Q_k(s_k - s)} \\ &\rightarrow (\Omega + \epsilon)/(\Omega - \hat{t}), \quad \text{as } k \rightarrow +\infty. \end{aligned}$$

With the above curvature bound and the injectivity radius estimates coming from  $\kappa$ -noncollapsing, one can apply Hamilton's compactness theorem (cf [42] or Theorem 4.1.5 in [19]) to get a subsequence of  $\hat{g}_{i\bar{j}}^{(k)}(\hat{t})$  which converges in the  $C_{loc}^\infty$  topology to a limit metric  $g_{i\bar{j}}^{(\infty)}(t)$  in the Cheeger sense on  $(X^n, J_\infty)$  for some complex structure

$J_\infty$  such that  $g_{i\bar{j}}^{(\infty)}(t)$  is a solution to the KRF with  $t \in (-\infty, \Omega)$  and its curvature satisfies the bound

$$|Rm^{(\infty)}| \leq \Omega/(\Omega - t)$$

everywhere on  $X_\infty^n \times (-\infty, \Omega)$  with the equality somewhere at  $t = 0$ .

**Type II:** Take a sequence  $S_k \rightarrow 1$  and pick space-time points  $(x_k, s_k)$  such that, as  $k \rightarrow +\infty$ ,

$$Q_k(S_k - s_k) = \max_{x \in X, s \leq S_k} (S_k - s) |\hat{R}m|(x, s) \rightarrow +\infty,$$

where again we denote by  $Q_k = |\hat{R}m|(x_k, s_k)$ . Now translate the time and dilate the solution as before to get

$$\hat{g}_{i\bar{j}}^{(k)}(\hat{t}) = Q_k \hat{g}_{i\bar{j}}(s_k + Q_k^{-1} \hat{t}),$$

which is a solution to the KRF and satisfies the curvature bound

$$\begin{aligned} |\hat{R}m^{(k)}| &= Q_k^{-1} |\hat{R}m(\hat{g})| \leq \frac{(S_k - s_k)}{(S_k - s)} \\ &= \frac{Q_k(S_k - s_k)}{Q_k(S_k - s_k) - \hat{t}} \quad \text{for } \hat{t} \in [-Q_k s_k, Q_k(S_k - s_k)]. \end{aligned}$$

Then as before, by applying Hamilton's compactness theorem, there exists a subsequence of  $\hat{g}_{i\bar{j}}^{(k)}(\hat{t})$  which converges in the  $C_{loc}^\infty$  topology to a limit metric  $g_{i\bar{j}}^{(\infty)}(t)$  in the Cheeger sense on a limiting complex manifold  $(X_\infty^n, J_\infty)$  such that  $g_{i\bar{j}}^{(\infty)}(t)$  is a complete solution to the KRF with  $t \in (-\infty, +\infty)$ , and its curvature satisfies

$$|Rm^{(\infty)}| \leq 1$$

everywhere on  $X_\infty^n \times (-\infty, +\infty)$  and the equality holds somewhere at  $t = 0$ .  $\square$

*Remark 7.1.* The injectivity radius bound needed in Hamilton's compactness theorem is satisfied due to the "Little Loop Lemma", which is a consequence of Perelman's  $\kappa$ -noncollapsing theorem.

Thanks to Perelman's monotonicity of  $\mu$ -entropy and the uniform scalar curvature bound in Theorem 6.1, we can say more about the singularity models in Theorem 7.1.

First of all, the following result on Type I singularity models of KRF (2.1) is well-known (cf. [66]).

**Theorem 7.2.** *Let  $\tilde{g}_{i\bar{j}}(s)$  be a Type I solution to KRF (2.1) on  $X^n \times [0, 1)$  and  $g_{i\bar{j}}(t)$  be the corresponding nonsingular solution to NKRF (2.5) on  $X^n \times [0, \infty)$ . Then there exists a sequence  $\{t_k\} \rightarrow \infty$  such that  $g_{i\bar{j}}^{(k)}(t) =: g_{i\bar{j}}(t + t_k)$  converges in the Cheeger sense to a gradient shrinking Kähler-Ricci soliton  $g^\infty(t)$  on  $(X^n, J_\infty)$ , where  $J_\infty$  is a certain complex structure on  $X^n$ , possibly different from  $J$ .*

**Proof.** This is a consequence of Theorem 7.1, and the fact that every compact Type I singularity model is necessarily a shrinking gradient Ricci soliton (see [66], [67] or p.662 of [61]; also Corollary 1.2 in [14]).  $\square$

Next, for Type II solutions to the KRF, we have the following two results. These results were known to R. Hamilton and the author [17] back in 2004<sup>5</sup>, and also observed independently by Ruan-Zhang-Zhang [64] (see also [25]).

**Theorem 7.3.** *Let  $g_{i\bar{j}}(t)$  be a singular solution to NKRF (2.5) on  $X^n \times [0, \infty)$ . Then there exists a sequence  $\{t_k\} \rightarrow \infty$  and rescaled solution metrics  $g^{(k)}(t)$  to KRF such that  $(X^n, J, g^{(k)}(t))$  converges in the Cheeger sense to some noncompact limit  $(X_\infty^n, J_\infty, g_\infty(t))$ ,  $-\infty < t < \infty$ , with the following properties:*

- (i)  $g_\infty(t)$  is Calabi-Yau (i.e., Ricci flat Kähler);
- (ii)  $|Rm|_{g_\infty(t)}(x, t) \leq 1$  everywhere and with equality somewhere at  $t = 0$ ;
- (iii)  $(X_\infty^n, g_\infty(t))$  has maximal volume growth: for any  $x_0 \in X_\infty^n$  there exists a positive constant  $c > 0$  such that

$$\text{Vol}(B(x_0, r)) \geq cr^{2n}, \quad \text{for all } r > 0.$$

**Proof.** This is an immediate consequence of Theorem 7.1 and Theorem 6.1 (i). Indeed, Theorem 7.1 implies the existence of a noncompact Type II singularity model  $(X_\infty^n, J_\infty, g_\infty(t))$  satisfying property (ii). Property (iii) follows from the factor that the  $\kappa$ -noncollapsing property for KRF or NKRF in Theorem 4.2 is dilation invariant, hence (4.17) and (4.18) holds for each rescaled solution on larger and larger scales for the same  $\kappa > 0$ , hence the maximal volume growth in the limit of dilations. Finally, for property (i), note that the scalar curvature  $R$  of  $g_{i\bar{j}}(t)$  is uniformly bounded on  $X \times [0, \infty)$  by Theorem 6.1 and the rescaling factors go to infinite, so we have  $R^\infty = 0$  everywhere in the limit of dilations. On the other hand, since  $g_{i\bar{j}}^\infty(t)$  is a solution to KRF,  $R^\infty$  satisfies the evolution equation

$$\frac{\partial}{\partial t} R^\infty = \Delta R^\infty + |Rc^\infty|^2.$$

Thus, we have  $|Rc^\infty|^2 = 0$  everywhere hence  $g_\infty$  is Ricci-flat.  $\square$

**Theorem 7.4.** *Let  $X^2$  be a Del Pezzo surface (i.e., a Fano surface) and let  $g_{i\bar{j}}(t)$  be a singular solution to NKRF (2.5) on  $X^2 \times [0, \infty)$ . Then the Type II limit space  $(X_\infty^2, J_\infty, g_\infty)$  in Theorem 7.3 is a non-compact Calabi-Yau space satisfies the following properties:*

- (a)  $|Rm|_{g_\infty} \leq 1$  everywhere on  $X_\infty^2$  and with equality somewhere;
- (b)  $(X_\infty^2, g_\infty)$  has maximal volume growth: for any  $x_0 \in X_\infty^2$  there exists a positive constant  $c > 0$  such that

$$\text{Vol}(B(x_0, r)) \geq cr^4, \quad \text{for all } r > 0;$$

- (c)  $\int_{X_\infty^2} |Rm(g_\infty)|^2 dV_\infty < \infty$ .

**Proof.** Clearly, we only need to verify property (c). But this follows from the facts the integral

$$\int_{X^2} |Rm|^2(x, t) dV_t$$

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<sup>5</sup>Theorem 7.3 and Theorem 7.4 were observed by Hamilton and the author during the IPAM conference “Workshop on Geometric Flows: Theory and Computation” in February, 2004.



is dilation invariant in complex dimension  $n = 2$  (real dimension 4); that it differs from  $\int_X R^2 dV_t$  up to a constant depending only on the Kähler class of  $g(0)$  and the Chern classes  $c_1(X)$  and  $c_2(X)$  (cf. Proposition 1.1 in [7]); and that, before the dilations,  $\int_X R^2 dV_t$  is uniformly bounded for all  $t \in [0, \infty)$  by the uniform scalar curvature bound in Theorem 6.1 (i).  $\square$

*Remark 7.2.* The work of Bando-Kasue-Nakajima [5] implies that the limiting Calabi-Yau surfaces in Theorem 7.4 are asymptotically locally Euclidean (ALE) of order at least 4.

*Remark 7.3.* Kronheimer [46] has classified ALE Hyper-Kähler surfaces (i.e., simply connected ALE Calabi-Yau surfaces).

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